THE MECHANICS OF DEFORMABLE BODIES

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THE MECHANICS OF DEFORMABLE BODIES

Being Volume II of

"INTRODUCTION TO THEORETICAL PHYSICS"

BY

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AUTHOR'S PREFACE

THE problem which I have proposed in the present volume and the method adopted for its solution call for essentially the same comments as those made in the preface of the preceding volume, which deals with General Mechanics. It is hoped that this little book will help the reader to form clearly defined concepts and to realize their relatedness, since here the mechanics of deformable bodies is developed as an inherently necessary consequence of general mechanics. After reading these two volumes he should be able not only to study the more detailed text-books and the advanced treatises with complete understanding, but also to supplement them by independent and penetrating researches of his own.

By occasionally referring to the context of the theorems and equations derived in the first volume it has been possible to abbreviate and simplify the present account. Such references are indicated by the Roman symbol I. Thus I, (155) denotes equation (155) of the volume on General Mechanics. I. § 49 denotes § 49 of that volume. A certain amount of space has also been saved by omitting formulæ and intermediate calculations which can be added by any reader who has received a mathematical training.

An alphabetical list of the definitions used and the most important theorems is appended, which will facilitate reference.

MAX PLANCK.

Berlin, Grunewald,
February 1919.

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In this new edition a few misprints have been corrected and a few small additions have been made.

MAX PLANCK.

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INTRODUCTION

§ 1. A DEFORMABLE body, as distinguished from a rigid body, is one which is susceptible of a change of form either as a whole or in any of its parts. Strictly speaking, all bodies in Nature are deformable, for there are no motions which are not accompanied by greater or smaller changes of form in the bodies participating in them. But in many cases—for example, in that of the pendulum, the lever, the top—it is sufficient, for a first approximation to reality, to assume the bodies in question as rigid. The motions of rigid bodies are dealt with in general mechanics. Here we shall deal with those motions for whose peculiarities the deformations are of characteristic importance. Hence we must now refine our assumptions about the constitution of material bodies a degree further. We do this in the first place by assuming everywhere that the space occupied by a body is *continuously* filled with matter. It is true that this assumption, like that of rigidity, is only an ideal abstraction, and is never rigorously fulfilled in Nature, since, strictly speaking, all bodies have an atomic structure. But as a first approximation to reality we find that this simplifying assumption is likewise completely sufficient here. For, just as it would be superfluous and inexpedient in establishing the elementary laws of levers to take into account the elastic bending of the lever that actually always occurs, so we should be adopting clumsy methods in investigating the fundamental laws of sound waves or of the flow of liquids if we wished at the outset to revert to the molecules or even the invariable atoms of the bodies in question, particularly as the latter undoubtedly again represent

an ideal abstraction. Nature does not allow herself to be exhaustively expressed in human thought.

It is the most important and at the same time the most difficult task of the theoretical physicist, when finding a mathematical formulation of the problem which he is attacking, to introduce just those simplifying assumptions which are of characteristic importance for the properties which interest him in the physical process which he is investigating, and to neglect all influences of a lower order of magnitude which can produce no essential change in the main result and which enter into the argument as mathematical ballast. An important and inevitable postulate is that the different hypotheses introduced for dealing with different problems should be compatible with each other. For otherwise the physical picture of the world would lose its uniformity, and in some circumstances we should get two mutually contradictory answers to a definite question.

For our present purpose it seems advantageous to divide the whole subject under discussion into *three* distinct parts. In the first we derive the general laws of motion of continuously extended bodies, no account whatsoever being taken of their aggregate state, whereas in the second and third parts we deal with applications to the most important kinds of motion, according as they are associated with infinitely small or with finite deformations.

PART ONE

GENERAL LAWS OF MOTION OF A CONTINUOUSLY EXTENDED BODY

CHAPTER I

LAWS OF KINEMATICS

§ 2. As in the general mechanics of material points and rigid bodies, so here, where we are dealing with deformable bodies, we first consider the properties of the motions in themselves, without inquiring into their causes, and we endeavour to represent them fully in mathematical terms. A motion of a material body is completely determined when, and only when, the motions of all the material points of which it may be imagined to be composed are known, or if—what comes to the same thing—the position of every one of these material points is given as a function of the time. To characterize a definite material point of a body we shall fix our attention on the state of the body at the time $t = 0$, and for this definite state we shall assign to every material point of the body its three co-ordinates a, b, c referred to a stationary rectangular and right-handed co-ordinate system (I, § 16). These three quantities a, b, c are then also to serve to characterize the point for the following times t, when the co-ordinates of the material point transform from a, b, c to x, y, z; in a certain sense they give the point a name, by means of which it can be discovered again at any time. The whole motion of the body is then determined in all its details if for each of the material points a, b, c of which the body is composed, x, y, z are given as functions of t, that is, if:

$$\left.\begin{aligned} x &= f(a, b, c, t) \\ y &= \phi(a, b, c, t) \\ z &= \psi(a, b, c, t) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

where f, ϕ, ψ denote certain uniform and finite functions of a, b, c, t. We also assume them to be continuous by assuming that the body does not divide into separate pieces in the course of the motion. In accordance with our above convention, we have for $t = 0$:

$$x = a,\ y = b,\ z = c \qquad . \quad . \quad . \quad . \qquad (1a)$$

In view of the abundance of the possibilities contained in (1), it is advantageous to fix on a perfectly definite interval of time t at first, or, in other words, to restrict ourselves to considering the change * (*Veränderung*) which occurs in the body in the time from $t = 0$ to $t = t$, where t has any constant value. We may then omit t entirely from the expressions (1) and have, more simply:

$$\left.\begin{aligned} x &= f(a, b, c) \\ y &= \phi(a, b, c) \\ z &= \psi(a, b, c) \end{aligned}\right\} \qquad . \quad . \quad . \quad . \qquad (2)$$

These equations denote the transition of a body from a given "initial position" to a definite "final position," in the process of which the material point (a, b, c) exchanges its position (a, b, c) for the position (x, y, z); for this reason the quantities:

$$x - a = u,\ y - b = v,\ z - c = w \qquad . \quad . \qquad (3)$$

are called the components of its "displacement."

If we solve the equations (2) for a, b, c, we obtain a, b, c as certain functions of x, y, z, whose values give the answer to the question as to where that material point which has the co-ordinates x, y, z *after* the change was situated *before* the change. We also assume these functions, which may be interpreted as representing the opposite change to that in question, as uniform, finite and continuous.

* The word change will often be used in the sequel as an abbreviation for change of position. When the mathematical aspect is to be stressed we shall find it convenient to use the word transformation.

§ 3. As our first example we shall consider the general case of a *linear change* :

$$\left.\begin{aligned} x &= \lambda_0 + \lambda_1 a + \lambda_2 b + \lambda_3 c \\ y &= \mu_0 + \mu_1 a + \mu_2 b + \mu_3 c \\ z &= \nu_0 + \nu_1 a + \nu_2 b + \nu_3 c \end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

The symbols for the constants have been so chosen that the letters λ, μ, ν correspond with the letters x, y, z, and the figures 1, 2, 3 with the letters a, b, c. The quantities λ_0, μ_0, ν_0 give the displacement of the material point which was situated at the origin of co-ordinates before the change.

Solved for a, b, c the equations (4) run :

$$\left.\begin{aligned} a &= \lambda'_1(x - \lambda_0) + \mu'_1(y - \mu_0) + \nu'_1(z - \nu_0) \\ b &= \lambda'_2(x - \lambda_0) + \mu'_2(y - \mu_0) + \nu'_2(z - \nu_0) \\ c &= \lambda'_3(x - \lambda_0) + \mu'_3(y - \mu_0) + \nu'_3(z - \nu_0) \end{aligned}\right\}. \quad (5)$$

where :

$$\lambda'_1 = \frac{[\lambda_1]}{D}, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the abbreviation D being used for the so-called functional determinant :

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} = D \quad . \quad . \quad . \quad . \quad (7)$$

and the minor of an element λ_1 in the expression for the determinant being denoted thus :

$$\mu_2\nu_3 - \mu_3\nu_2 = [\lambda_1], \text{ etc.} \quad . \quad . \quad . \quad . \quad (8)$$

Since we assume all constants, both accented and unaccented, to be finite, the functional determinant D cannot be equal to zero; this allows us to deduce the sign of D immediately. For the change does not occur suddenly, but gradually in a finite time t. Thus the constants λ, μ, ν of the change and with them the determinant D are to be regarded as continuous functions of t. Now, before the change—that is, for $t = 0$—we have :

$$\lambda_0 = 0, \ \lambda_1 = 1, \ \lambda_2 = 0, \ \lambda_3 = 0, \text{ etc.} \quad . \quad . \quad (9)$$

and accordingly $D = 1$. Hence in the course of time the determinant D, starting from the value 1, changes continuously without ever becoming zero. It follows from this that D is always positive :

$$D > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

§ 4. As a special case of a linear change we have a translation (I, § 102), in which all the material points of the body undergo displacements which are all in the same direction and all of the same amount, and this amount may be arbitrarily great. For in this case we have by (3) :

$$x - a = \text{const.}, \quad y - b = \text{const.}, \quad z - c = \text{const.}$$

that is, a special case of (4).

Another special case of a linear change is given by a rotation (I, § 101) about any axis through any arbitrarily great angle. To prove this we shall set up the equations for an arbitrary finite rotation. For this purpose we shall assume, besides the fixed co-ordinate system in space, whose origin we take in the axis of rotation, a second co-ordinate system which is fixed in the body—that is, is movable in space. We choose the latter system in such a way that the corresponding axes of the two systems coincide before the change. After the change each two axes, one from each system, form certain angles with each other, whose direction cosines we shall call α_1, β_1, γ_1, α_2, β_2, γ_2, α_3, β_3, γ_3, where the letters α, β, γ refer to the axes of the first system and the suffixes 1, 2, 3 refer to the axes of the second system.

Now before the change some material point of the body has the co-ordinates a, b, c with reference to each of the two systems. After the change, however, this same material point has the co-ordinates x, y, z with respect to the first system, but still the co-ordinates a, b, c with respect to the second system. Consequently the relationship between x, y, z and a, b, c is the same as that for the transformation of the co-ordinates of a point

in space from one co-ordinate system to another which is characterized by the nine given direction cosines, or by (I, (329)):

$$\left.\begin{aligned} x &= \alpha_1 a + \alpha_2 b + \alpha_3 c \\ y &= \beta_1 a + \beta_2 b + \beta_3 c \\ z &= \gamma_1 a + \gamma_2 b + \gamma_3 c \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

which is again a special case of (4).

If to this rotation (11) we add a translation whose displacement components are λ_0, μ_0, ν_0, we obtain the most general change which a rigid body is susceptible of; it is expressed by the equations:

$$\left.\begin{aligned} x &= \lambda_0 + \alpha_1 a + \alpha_2 b + \alpha_3 c \\ y &= \mu_0 + \beta_1 a + \beta_2 b + \beta_3 c \\ z &= \nu_0 + \gamma_1 a + \gamma_2 b + \gamma_3 c \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

But this change, too, is also a special case of the general linear change (4), since the twelve constants which characterize it are not independent of each other. Hence if we inquire under what condition a linear change (4) of a body occurs unaccompanied by a deformation of the body, the answer is that the same relations between the coefficients λ, μ, ν of the change must hold as hold between the corresponding coefficients of (12). By I, (331) and (332) there are six such relations—namely three of the form:

$$\lambda_1^2 + \mu_1^2 + \nu_1^2 = 1, \text{ and so forth} \quad . \quad . \quad (13)$$

and three of the form:

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0, \text{ and so forth} \quad . \quad . \quad (14)$$

These relationships simultaneously contain a series of further relationships, as, for example, the six which arise from (13) and (14) by permuting the letters λ, μ, ν with the figures 1, 2, 3 (I, (333) and (334)); further, the relation (I, (461)):

$$D = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and the nine relationships (I, (462)) of the form :

$$\lambda_1 = \mu_2\nu_3 - \mu_3\nu_2 = [\lambda_1] = \lambda'_1, \text{ and so forth} \quad . \quad (16)$$

On account of these relationships the accented coefficients in the equations (5) for the reverse change assume the similarly designated unaccented values.

§ 5. We now revert to the general case of a linear change and proceed to write down some of its properties. In the first place, it is easy to show that all the material points which lie in one plane before the change also lie in one plane after the change. For the material points (a, b, c) which lie in a plane satisfy a linear equation :

$$Aa + Bb + Cc + D = 0 \quad . \quad . \quad . \quad (17)$$

After the change the position of each of these points will be given by the values of x, y, z which follow from (4). Hence if the a, b, c 's satisfy the equation (17), the x, y, z 's fulfil a condition which is obtained by eliminating a, b, c most conveniently by substituting the values (5) in (17). But this, again, gives a linear equation. Consequently the points (x, y, z) also lie in one plane.

As a direct result of this theorem of the conservation of planes we have the theorem of the conservation of straight lines, since a straight line is given by the intersection of two planes, and also the theorem of the conservation of the order of any surface, since the order-number of a surface is determined by the number of points in which it intersects a straight line.

Parallel lines also remain preserved in a linear change. For if two material planes, which were parallel before the change, were no longer parallel after the change, the line of intersection of the planes which would then occur in finite regions would be formed by those material points (a, b, c), which, before the change, were at infinity—that is, for which, although the x, y, z 's are finite, at least some of the a, b, c 's would be infinite. But by equations (5) this is impossible.

By combining the conservation of planes and the

conservation of parallelism, it follows that all parallelepipeds also remain preserved. But their angles and volumes may change. We shall now calculate the change of volume of any parallelepiped cut out of a body, and shall take the parallelepiped as characterized by any four arbitrarily chosen corner points $a_1, b_1, c_1, \ldots a_4, b_4, c_4$. Before the change the volume of this parallelepiped is given by:

$$V = \pm \begin{vmatrix} a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ a_3 & b_3 & c_3 & 1 \\ a_4 & b_4 & c_4 & 1 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (18)$$

but after the change by:

$$V' = \pm \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (19)$$

where the x, y, z 's are linked by the equations (4) with the a, b, c 's bearing similar suffixes.

The relationship between V and V' is obtained most simply by multiplying the determinant (18) by the functional determinant (7), which may be written in the following way as a determinant of the fourth order:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_0 \\ \mu_1 & \mu_2 & \mu_3 & \mu_0 \\ \nu_1 & \nu_2 & \nu_3 & \nu_0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = D \quad . \quad . \quad . \quad . \quad (20)$$

By the theorem of multiplication of determinants the product of the determinants (18) and (20) is again a determinant of the fourth order, whose individual terms are obtained if we combine any one row of the one determinant with any one row of the other determinant by multiplying together the similarly situated terms of the two rows and adding together the products

so obtained. So the first term of the first row in the resultant determinant runs :

$$a_1 . \lambda_1 + b_1 . \lambda_2 + c_1 . \lambda_3 + 1 . \lambda_0$$

and the second term of the first row :

$$a_1 . \mu_1 + b_1 . \mu_2 + c_1 . \mu_3 + 1 . \mu_0$$

and the fourth term of the first row :

$$a_1 . 0 + b_1 . 0 + c_1 . 0 + 1 . 1$$

and so forth. In view of (4) the whole determinant assumes exactly the same form as the determinant (19). Accordingly we have :

$$V . D = V' \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and we note that the quantities V, V' and D are all three positive.

Since this relationship is independent of the magnitude and form of the parallelepiped which we are considering, and hence, in particular, also holds for infinitely small parallelepipeds, we may immediately generalize equation (21) to apply to any arbitrary volume cut out of the body and so arrive at the theorem : in a linear change the volumes of all parts of the body change uniformly, the ratio of the volume of any part of the body after the change to the volume of the same part before the change being equal to the functional determinant. It is in agreement with this that, by (15), for every change of a rigid body the functional determinant is equal to 1.

The ratio of the change of volume to the original volume is called the " dilatation " of the volume. Hence the volume dilatation in a linear change :

$$\frac{V' - V}{V} = D - 1 \quad . \quad . \quad . \quad . \quad (22)$$

is positive or negative according as expansion or contraction occurs.

§ 6. If we subject a body to two or more linear changes in succession, the resultant change again represents a linear change. This result emerges at once if we consider the relationship between the co-ordinates of any material point after the final change and its initial co-ordinates (a, b, c). If the point arrives after the first change at the point (x, y, z), after the second at the point (x', y', z') and so forth, and after the last change at the point (x^*, y^*, z^*), we obtain the relationships between (a, b, c), and (x^*, y^*, z^*) by eliminating the intermediate co-ordinates, and these relationships are obviously again linear equations.

Conversely, every linear change may be resolved into a succession of linear changes, but, of course, the order in which these changes are effected is not a matter of indifference for their physical significance, and it suggests itself to us to perform this resolution in such a way that the character of the individual changes remains as simple and as easily pictured physically as possible. For in this way it will be possible to reduce the general case (4) of a linear change to a number of simpler changes that can easily be followed, and hence at the same time to bring out clearly the physical significance of the constants λ, μ, ν of the change.

The first step in this direction is taken by reducing the change (4) to another in which the material point $a = 0, b = 0, c = 0$ retains its position. This is achieved simply by giving the body a *translation* with the components λ_0, μ_0, ν_0 and so bringing this point to its final position. There then still remains a so-called "homogeneous change," whose general form is:

$$\left.\begin{aligned} x &= \lambda_1 a + \lambda_2 b + \lambda_3 c \\ y &= \mu_1 a + \mu_2 b + \mu_3 c \\ z &= \nu_1 a + \nu_2 b + \nu_3 c \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (23)$$

We shall now consider the latter in some detail.

Let us fix our attention on all those material points which, before the change, lie on a spherical surface de-

scribed with the arbitrary radius R about the origin of co-ordinates as centre, that is:

$$a^2 + b^2 + c^2 = R^2 \quad . \quad . \quad . \quad . \quad (24)$$

After the change these points lie, by § 5, on a certain surface of the second order, and since no point moves off to infinity, this surface must be an ellipsoid whose centre is the origin of co-ordinates, as we recognize at once if in (24) we replace the quantities a, b, c by x, y, z by means of (23). Those material straight lines which before the change coincide with the co-ordinate axes will not, in general, be perpendicular to one another after the change. But we may assert that after the change they form a triplet of conjugate diameters with respect to the ellipsoid—that is, the tangential plane drawn through the end-point of every diameter of the ellipsoid is parallel to the plane drawn through the other two diameters. For in the first place the three co-ordinate axes, like any three mutually perpendicular straight lines, form a triplet of conjugate diameters with respect to the sphere, and, secondly, the property of constituting a conjugate triplet belongs to those properties which are not destroyed by a linear change, since both the tangential plane and parallelism remain preserved.

Now, among all the triplets of conjugate diameters to the ellipsoid there is a perfectly definite triplet which consists of right angles only; this triplet consists of the axes of the ellipsoid; from this it follows that the three material straight lines which coincide with the axes of the ellipsoid after the change were also mutually perpendicular before the change, since they were then conjugate with respect to the sphere. In other words, there are three definite material straight lines which are mutually perpendicular both before and after the change.

With the help of this theorem *every homogeneous linear change* can be resolved into (1) a simple *rotation* about the origin of co-ordinates, which is so arranged that it brings the three straight lines just mentioned into their new

directions, and (2) a certain linear change which has the property of making three mutually perpendicular straight lines preserve their directions; we call this latter change a *dilatation in three mutually perpendicular directions.*

§ 7. To investigate the properties of a dilatation in three mutually perpendicular directions more closely, we choose the co-ordinate axes to lie in these three directions—the so-called "dilatation axes"—and determine how the general equations (23) of a homogeneous linear change become simplified with the assumptions made. If the direction of the x-axis is to be preserved, we must have $y = 0$ and $z = 0$, for $b = 0$ and $c = 0$—that is $\mu_1 = 0$ and $\nu_1 = 0$. Corresponding results hold for the other two axes. Hence we obtain as the general expression for a dilatation in the directions of the three co-ordinate axes the equations:

$$x = \lambda_1 a, \ y = \mu_2 b, \ z = \nu_3 c \quad . \quad . \quad . \quad (25)$$

By (7) and (10) the functional determinant is:

$$D = \lambda_1 \mu_2 \nu_3 > 0 \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and each of the three coefficients is individually positive, since the co-ordinate axes do not reverse their directions.

But the co-ordinate axes are also, at least in the general case, the only directions which are not affected by the change. For the material points which before the change lie on a straight line with direction ratios $\alpha : \beta : \gamma = a : b : c$ will form, after the change, a straight line whose direction ratios are $x : y : z = \lambda_1 \alpha : \mu_2 \beta : \nu_3 \gamma$.

Whereas, by (22), the volume dilatation is given by:

$$D - 1 = \lambda_1 \mu_2 \nu_3 - 1 \quad . \quad . \quad . \quad . \quad (27)$$

we obtain the dilatation of a straight line if we divide the change in the distance between two points of this straight line by the original distance between them. Thus the dilatations of the three axes are:

$$\frac{x-a}{a} = \lambda_1 - 1, \ \frac{y-b}{b} = \mu_2 - 1, \ \frac{z-c}{c} = \nu_3 - 1 \quad (28)$$

They are usually called the "principal dilatations" and give a clear physical interpretation to the three coefficients of the change.

In the special case $\lambda_1 = \mu_2 = \nu_3$ the three principal dilatations become equal to each other, positive or negative, all the straight lines retain their directions, the dilatation axes become indeterminate and the body undergoes a uniform extension or contraction in all directions, each of its parts remaining similar to itself in the process.

§ 8. Now that we have seen that every linear displacement can be regarded as produced by a translation, a rotation and a dilatation in three mutually perpendicular directions, our next task is actually to derive these individual operations for a definite change given by the equations (4), or, in other words, to calculate from the given coefficients λ, μ, ν the corresponding individual changes. For this purpose we shall first convince ourselves that in order to calculate all the required quantities there are exactly as many equations available as there are unknown quantities. Actually, the equations (4) contain twelve mutually independent constants λ, μ, ν, which we regard as given, and the number of quantities characteristic for the individual changes is equally great—namely, three for the translation, three for the rotation, and six for the dilatation in three mutually perpendicular directions, since three quantities are required to determine the directions of the dilatation axes and three other quantities to determine principal dilatations.

As was shown in § 6, we obtain as the components of translation simply the values λ_0, μ_0, ν_0, and after they have been detached the equations (23) for a homogeneous change remain. In order to connect the nine coefficients of the latter with the characteristics of rotation and dilatation we first imagine the body to be subjected to a dilatation in three mutually perpendicular directions and then to a rotation about the origin of co-ordinates, and we then calculate the change of position which a definite material point (a, b, c) experiences as a result of

all these motions. Let the directions of the three dilatation axes, which do not of course in general coincide with the co-ordinate axes, be denoted by the nine direction cosines α_1, β_1, γ_1, α_2, β_2, γ_2, α_3, β_3, γ_3, the constants of the dilatation by l_0, m_0, n_0, since the letters λ, μ, ν in (23) already occur in (23) with another meaning. In order to be able to apply the equations (25) for the dilatation here we first refer the material point (a, b, c) to the dilatation axes as co-ordinate axes, so that its co-ordinates become:

$$\left.\begin{aligned} \alpha_1 a + \beta_1 b + \gamma_1 c &= \xi \\ \alpha_2 a + \beta_2 b + \gamma_2 c &= \eta \\ \alpha_3 a + \beta_3 b + \gamma_3 c &= \zeta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (29)$$

We then perform the dilatation and from (25) we then obtain as the co-ordinates of the same material point, after the change, with respect to the dilatation axes:

$$\xi' = l_0\xi,\ \eta' = m_0\eta,\ \zeta' = n_0\zeta \quad . \quad . \quad . \quad (30)$$

If the rotation is now performed, let the dilatation axes pass from the directions denoted by the α, β, γ's to certain other directions to which the direction cosines α'_1, β'_1, γ'_1, α'_2, β'_2, γ'_2, α'_3, β'_3, γ'_3 correspond. After the rotation has been performed our material point then has the co-ordinates ξ', η', ζ' with respect to the three directions $(\alpha', \beta', \gamma')$, since its position with respect to the dilatation axes is not changed by the rotation. But with respect to the original co-ordinate system which is fixed in space it has the co-ordinates x, y, z, because when the rotation has been completed the total change has been performed in the body. So we have:

$$\left.\begin{aligned} x &= \alpha'_1\xi' + \alpha'_2\eta' + \alpha'_3\zeta' \\ y &= \beta'_1\xi' + \beta'_2\eta' + \beta'_3\zeta' \\ z &= \gamma'_1\xi' + \gamma'_2\eta' + \gamma'_3\zeta' \end{aligned}\right\} \quad . \quad . \quad . \quad (31)$$

These equations furnish us with the required bridge and we have now reduced the co-ordinates x, y, z to terms of the co-ordinates a, b, c. In them we now need

only express ξ', η', ζ' by means of (30) in terms of ξ, η, ζ and then the latter, by (29), in terms of a, b, c. If we do this and then identify the three equations so obtained with the general equations (23) of a linear homogeneous change, we obtain the following nine relationships :

$$\left.\begin{aligned}
l_0\alpha_1\alpha'_1 + m_0\alpha_2\alpha'_2 + n_0\alpha_3\alpha'_3 &= \lambda_1 \\
l_0\beta_1\alpha'_1 + m_0\beta_2\alpha'_2 + n_0\beta_3\alpha'_3 &= \lambda_2 \\
l_0\gamma_1\alpha'_1 + m_0\gamma_2\alpha'_2 + n_0\gamma_3\alpha'_3 &= \lambda_3 \\
l_0\alpha_1\beta'_1 + m_0\alpha_2\beta'_2 + n_0\alpha_3\beta'_3 &= \mu_1 \\
l_0\beta_1\beta'_1 + m_0\beta_2\beta'_2 + n_0\beta_3\beta'_3 &= \mu_2 \\
l_0\gamma_1\beta'_1 + m_0\gamma_2\beta'_2 + n_0\gamma_3\beta'_3 &= \mu_3 \\
l_0\alpha_1\gamma'_1 + m_0\alpha_2\gamma'_2 + n_0\alpha_3\gamma'_3 &= \nu_1 \\
l_0\beta_1\gamma'_1 + m_0\beta_2\gamma'_2 + n_0\beta_3\gamma'_3 &= \nu_2 \\
l_0\gamma_1\gamma'_1 + m_0\gamma_2\gamma'_2 + n_0\gamma_3\gamma'_3 &= \nu_3
\end{aligned}\right\} \quad . \quad . \quad . \quad (32)$$

whose method of construction is perfectly clear. If λ, μ, ν are given we obtain from these nine relationships the values of the nine unknowns contained in them.

§ 9. In performing the calculation we shall restrict ourselves to the special case where the *linear change is infinitely small.* This case is very approximately realized in many processes, particularly in many involving solid bodies, but it also always has a certain importance for finite changes in so far as a finite change in Nature always occurs in a finite time, and hence resolves into a series of infinitely small changes which occur in the successive infinitely small intervals of time.

In order to introduce in a convenient form the simplifications which occur in the case of an infinitely small change we find it expedient to use instead of the finite final co-ordinates x, y, z of a material point its infinitely small displacement components (3). The equations (4) of the change then become :

$$\left.\begin{aligned}
u &= \lambda_0 + \lambda a + \lambda_2 b + \lambda_3 c \\
v &= \mu_0 + \mu_1 a + \mu b + \mu_3 c \\
w &= \nu_0 + \nu_1 a + \nu_2 b + \nu c
\end{aligned}\right\} \quad . \quad . \quad . \quad (33)$$

where we have used the abbreviations :

$$\lambda_1 - 1 = \lambda,\ \mu_2 - 1 = \mu,\ \nu_3 - 1 = \nu \quad . \quad . \quad (34)$$

Now *all* the twelve coefficients λ, μ, ν in (33) are infinitely small.

On the other hand, of course, the principal dilatations (28) are infinitely small. We therefore set its values :

$$l_0 - 1 = l,\ m_0 - 1 = m,\ n_0 - 1 = n \quad . \quad . \quad (35)$$

Further, the direction cosines α, β, γ of the dilatation axes, whose values in themselves will in general clearly be finite, undergo only infinitely small changes as a result of the rotation, so that we can write :

$$\alpha'_1 = \alpha_1 + d\alpha_1,\ \beta'_1 = \beta_1 + d\beta_1, \text{ and so forth.}$$

If we introduce all these substitutions into the equations (32) and moreover take into account the relationships (I, (333), (334)) which hold in general for the direction cosines as well as those (I, (335), (336)) which hold between their differentials, these nine equations assume the following form :

$$\left.\begin{array}{l}
l\alpha_1^2 + m\alpha_2^2 + n\alpha_3^2 = \lambda \\
l\alpha_1\beta_1 + m\alpha_2\beta_2 + n\alpha_3\beta_3 - \zeta = \lambda_2 \\
l\alpha_1\gamma_1 + m\alpha_2\gamma_2 + n\alpha_3\gamma_3 + \eta = \lambda_3 \\
l\alpha_1\beta_1 + m\alpha_2\beta_2 + n\alpha_3\beta_3 + \zeta = \mu_1 \\
l\beta_1^2 + m\beta_2^2 + n\beta_3^2 = \mu \\
l\beta_1\gamma_1 + m\beta_2\gamma_2 + n\beta_3\gamma_3 - \xi = \mu_3 \\
l\alpha_1\gamma_1 + m\alpha_2\gamma_2 + n\alpha_3\gamma_3 - \eta = \nu_1 \\
l\beta_1\gamma_1 + m\beta_2\gamma_2 + n\beta_3\gamma_3 + \xi = \nu_2 \\
l\gamma_1^2 + m\gamma_2^2 + n\gamma_3^2 = \nu
\end{array}\right\} \quad . \quad . \quad (36)$$

where, by I, § 101, ξ, η, ζ denote the components of that infinitely small rotation which brings the dilatation axes from the directions (α, β, γ) into the directions $(\alpha', \beta', \gamma')$. Their values come out directly as :

$$\xi = \frac{\nu_2 - \mu_3}{2},\ \eta = \frac{\lambda_3 - \nu_1}{2},\ \zeta = \frac{\mu_1 - \lambda_2}{2} \quad . \quad . \quad (37)$$

whereas for the magnitude and direction of the dilatation the following six equations remain :

$$\left.\begin{aligned}
l\alpha_1^2 + m\alpha_2^2 + n\alpha_3^2 &= \lambda \\
l\beta_1^2 + m\beta_2^2 + n\beta_3^2 &= \mu \\
l\gamma_1^2 + m\gamma_2^2 + n\gamma_3^2 &= \nu \\
l\beta_1\gamma_1 + m\beta_2\gamma_2 + n\beta_3\gamma_3 &= \frac{\nu_2 + \mu_3}{2} \\
l\gamma_1\alpha_1 + m\gamma_2\alpha_2 + n\gamma_3\alpha_3 &= \frac{\lambda_3 + \nu_1}{2} \\
l\alpha_1\beta_1 + m\alpha_2\beta_2 + n\alpha_3\beta_3 &= \frac{\mu_1 + \lambda_2}{2}
\end{aligned}\right\} \quad . \quad . \quad (38)$$

According to this we can immediately determine by looking at the determinant D in (7) whether the change is restricted, except for the translation, simply to a dilatation in three mutually perpendicular directions or whether there is also a rotation associated with it. For if the determinant is symmetrical—that is, if when the columns are changed into rows and the rows into columns—it remains unaltered in appearance, ξ, η, and ζ are all equal to zero and no rotation occurs. But it is to be carefully remarked that only the dilatation axes do not rotate. Other straight lines, as we saw in § 7, will undergo changes of direction even in the case of a pure dilatation.

The opposite case, where the change is restricted, except for the translation, to a rotation is characterized by the fact that the principal dilatations l, m, n, vanish. Then, by (38) and (37)

$$\lambda = \mu = \nu = 0$$

$$\nu_2 = -\mu_3 = \xi,\ \lambda_3 = -\nu_1 = \eta,\ \mu_1 = -\lambda_2 = \zeta$$

and the equations (33) for the displacement components become :

$$\left.\begin{aligned}
u &= \lambda_0 - \zeta b + \eta c \\
v &= \mu_0 + \zeta a - \xi c \\
w &= \nu_0 - \eta a + \xi b
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (39)$$

which are identical with the general expressions (I, (348)) for the displacements of the points of a rigid body.

The problem now remains to calculate the principal dilatations l, m, n and the direction cosines (α, β, γ) from the equations (38). For this purpose we again use the general relationships between the direction cosines, this time in the form I, (331), (332). With their help we immediately get the following relationships from (38):

$$\left.\begin{aligned} \lambda\alpha_1 + \frac{\mu_1+\lambda_2}{2}\beta_1 + \frac{\lambda_3+\nu_1}{2}\gamma_1 &= l\alpha_1 \\ \frac{\mu_1+\lambda_2}{2}\alpha_1 + \mu\beta_1 + \frac{\nu_2+\mu_3}{2}\gamma_1 &= l\beta_1 \\ \frac{\lambda_3+\nu_1}{2}\alpha_1 + \frac{\nu_2+\mu_3}{2}\beta_1 + \nu\gamma_1 &= l\gamma_1 \end{aligned}\right\} \quad . \quad (40)$$

These three equations are linear and homogeneous in α_1, β_1, γ_1. Since these quantities cannot possibly be zero simultaneously, the determinant of the equations must vanish—that is, we must have:

$$\begin{vmatrix} \lambda - l & \frac{\mu_1+\lambda_2}{2} & \frac{\lambda_3+\nu_1}{2} \\ \frac{\mu_1+\lambda_2}{2} & \mu - l & \frac{\nu_2+\mu_3}{2} \\ \frac{\lambda_3+\nu_1}{2} & \frac{\nu_2+\mu_3}{2} & \nu - l \end{vmatrix} = 0 \quad . \quad . \quad (41)$$

a cubic equation in l. Since the index 1 is in no way distinguished in the coefficients of these equations, it follows that the same equation also holds for each of the other two principal dilatations m and n, or, in other words, that the three roots of this equation themselves represent the values of l, m, n, and hence are also always real. Which root is to be designated by l, m, n of course remains undetermined; we could, for example, agree that $l \geqq m \geqq n$ without restricting the generality of the problem. The knowledge of l, m, n then also allows us

to calculate, by (40), the corresponding α, β, γ's except for the common sign, which remains indefinite.

For the volume dilatation (22) we get in the case of an infinitely small change the following value from the functional determinant (7) if we take (34) into account and neglect infinitely small quantities of a higher order:

$$D - 1 = \lambda + \mu + \nu \quad . \quad . \quad . \quad . \quad (42)$$

Hence only the diagonal terms of the determinant contribute to the volume dilatation. We arrive at the same result of course if we reflect that the rotation does not come into consideration at all for the volume dilatation, so that by (27) and (35):

$$D - 1 = l_0 m_0 n_0 - 1 = l + m + n \quad . \quad . \quad (43)$$

This is the sum of the roots of the cubic equation in l—namely (41)—and so is equal to the coefficient of l in this equation:

$$l + m + n = \lambda + \mu + \nu. \quad . \quad . \quad . \quad (44)$$

as can also be seen directly by adding together the first three equations of (38).

§ 10. The resolution of a linear transformation into a translation, a rotation and a dilatation in three mutually perpendicular directions is not the only possible resolution, but recommends itself particularly on account of its simple physical significance; there are other also comparatively simple resolutions. For example, the infinitely small homogeneous transformation:

$$\left.\begin{aligned} u &= \lambda a + \lambda_2 b + \lambda_3 c \\ v &= \mu_1 a + \mu b + \mu_3 c \\ w &= \nu_1 a + \nu_2 b + \nu c \end{aligned}\right\} \quad . \quad . \quad . \quad (45)$$

may be regarded as produced by a dilatation in the directions of the three co-ordinate axes, the principal dilatations being λ, μ, ν, and, in addition, by six transformations which are represented by the equations $u = \lambda_2 b$, $v = 0$,

$w = 0$, $u = \lambda_3 c$, $v = 0$, $w = 0$, and so forth. The order in which these transformations are carried out is immaterial, since if it is changed only differences of a smaller order of magnitude are produced (cf. I, § 101, in this connection). Each of these six transformations has a simple meaning: in the first the displacements are independent of a and c and all take place in the direction of the x-axis—that is, every plane parallel to the xz-plane executes a simple translation, it displaces itself as a whole without rotation or deformation in the x-direction, and only the magnitude of the displacement varies from plane to plane. Such a transformation resembles the way in which the two halves of a pair of shears glide over each other, and is therefore called a shear. Accordingly we have the theorem that every infinitely small homogeneous transformation can be produced by means of a dilatation in the directions of the three co-ordinate axes and six shears of the planes parallel to the co-ordinate planes, the motion of the shears being in the direction of the axes.

In spite of its simple form, the theorem, however, in general has a comparatively complicated physical significance. If, for example, we consider the infinitely small rotation $u = -\zeta b$, $v = \zeta a$, $w = 0$, which is a special case of the general displacement (39), it could be regarded, in our present view, as produced by two successive shears—that is, by transformations which are connected with deformations of the body. The introduction of the latter is quite superfluous, since they of course cancel each other.

§ 11. The case of a linear transformation which was investigated in the last paragraph is not only of interest as a special simple case, but also has great importance for the most general case of an *arbitrary finite transformation*, which we now proceed to discuss by reverting to equations (2). The essential reason for this is that any arbitrary function can be regarded as a linear function within an infinitely small region of its variables.

For if we fix our attention on a definite material point

P_0, whose co-ordinates are a_0, b_0, c_0, the transformation transfers this point to the point x_0, y_0, z_0, where, by (2):

$$x_0 = f(a_0, b_0, c_0), \; y_0 = \phi(a_0, b_0, c_0), \; z_0 = \psi(a_0, b_0, c_0) \quad . \quad (46)$$

If we now further consider any material point situated in the immediate vicinity of P_0 and having the co-ordinates:

$$a = a_0 + a', \; b = b_0 + b', \; c = c_0 + c' \quad . \quad . \quad (47)$$

which we shall suppose transferred to the point:

$$x = x_0 + x', \; y = y_0 + y', \; z = z_0 + z' \quad . \quad . \quad (48)$$

by the transformation, then by our hypothesis a', b', c' are infinitely small, and consequently by (2) and (46), if we expand by Taylor's Theorem:

$$\left.\begin{aligned} x' &= \left(\frac{\partial x}{\partial a}\right)_0 a' + \left(\frac{\partial x}{\partial b}\right)_0 b' + \left(\frac{\partial x}{\partial c}\right)_0 c' \\ y' &= \left(\frac{\partial y}{\partial a}\right)_0 a' + \left(\frac{\partial y}{\partial b}\right)_0 b' + \left(\frac{\partial y}{\partial c}\right)_0 c' \\ z' &= \left(\frac{\partial z}{\partial a}\right)_0 a' + \left(\frac{\partial z}{\partial b}\right)_0 b' + \left(\frac{\partial z}{\partial c}\right)_0 c' \end{aligned}\right\} \quad . \quad . \quad (49)$$

By allowing a', b', c' to run through all the possible infinitely small values, we obtain from these equations the change in the infinitely small part of the body that surrounds the point P_0. Hence we can regard this change as produced by a finite translation with components:

$$x_0 - a_0 = u_0, \; y_0 - b_0 = v_0, \; z_0 - c_0 = w_0$$

which displaces a point P from the position (a, b, c) to the position:

$$a + u_0 = x_0 + a', \; b + v_0 = y_0 + b', \; c + w_0 = z_0 + c'$$

and, besides, by a finite linear homogeneous transformation (49) with the point P_0 as the stationary initial point, in the course of which the point P exchanges its co-ordinates (a', b', c') for the co-ordinates (x', y', z').

If we use the following abbreviations for the nine coefficients of the transformation (49):

$$\left.\begin{array}{lll}\left(\frac{\partial x}{\partial a}\right)_0 = \lambda_1, & \left(\frac{\partial x}{\partial b}\right)_0 = \lambda_2, & \left(\frac{\partial x}{\partial c}\right)_0 = \lambda_3 \\ \left(\frac{\partial y}{\partial a}\right)_0 = \mu_1, & \left(\frac{\partial y}{\partial b}\right)_0 = \mu_2, & \left(\frac{\partial y}{\partial c}\right)_0 = \mu_3 \\ \left(\frac{\partial z}{\partial a}\right)_0 = \nu_1, & \left(\frac{\partial z}{\partial b}\right)_0 = \nu_2, & \left(\frac{\partial z}{\partial c}\right)_0 = \nu_3\end{array}\right\} \quad . \quad . \quad (50)$$

we also arrive formally at the equations (23), and we can immediately assert all the consequences which we deduced earlier.

Hence when a body is transformed in any way, every individual infinitely small part of the body or every "element" of the body changes linearly, and the only difference as compared with the linear transformation previously considered consists in the fact that the coefficients (λ, μ, ν) of the transformation change from element to element. We shall now direct our attention to some of the most important theorems that result from this fact.

Even in the case of the most general transformation, infinitely small parallelepipeds remain preserved as such: only the angles can change to an arbitrary extent; in the same way, the order number of an infinitely small surface remains preserved. For example, an infinitely small sphere becomes transformed into an ellipsoid, the centre of the sphere becoming the centre of the ellipsoid. This gives rise to a result which appears strange at first sight. The surface of the body will always be formed by the same material points after the transformation as before the transformation. For every point that does not lie at the surface before the change may be regarded as the centre of a sphere that lies entirely in the interior and whose centre therefore also remains in the interior. The same holds, of course, for the converse transformation, and hence the result is proved.

The volume dilatation is determined precisely as in (22) by the functional determinant (7):

$$D = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (51)$$

where, for simplicity, we suppress the suffix 0 from now on. Here the value of D denotes the ratio of the volume of an element of the body that contains the material point (a, b, c) after the change to its volume before the change.

If we solve the equations (49) for a', b', c', we obtain linear homogeneous expressions with coefficients (λ', μ', ν'), whose values are given by (6):

$$\lambda'_1 = \frac{[\lambda_1]}{D} = \frac{\left[\frac{\partial x}{\partial a}\right]}{D} = \frac{\frac{\partial y}{\partial b}\frac{\partial z}{\partial c} - \frac{\partial y}{\partial c}\frac{\partial z}{\partial b}}{D}, \text{ and so forth} \quad . \quad (52)$$

On the other hand, however, we can also first solve the equations (2) for a, b, c, and only *then* make the substitutions (47) and (48) in the form of the expansions by Taylor's Theorem. In this way we get, analogously to (49):

$$\begin{aligned} a' &= \frac{\partial a}{\partial x}x' + \frac{\partial a}{\partial y}y' + \frac{\partial a}{\partial z}z' \\ b' &= \frac{\partial b}{\partial x}x' + \frac{\partial b}{\partial y}y' + \frac{\partial b}{\partial z}z' \\ c' &= \frac{\partial c}{\partial x}x' + \frac{\partial c}{\partial y}y' + \frac{\partial c}{\partial z}z' \end{aligned}$$

where now the coefficients must agree with the (λ', μ', ν')'s. This leads to the validity of the following transformation formulæ:

$$\frac{\partial a}{\partial x} = \frac{\left[\frac{\partial x}{\partial a}\right]}{D}, \quad \frac{\partial a}{\partial y} = \frac{\left[\frac{\partial y}{\partial a}\right]}{D}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

which refer the differential coefficients for independent x, y, z 's generally to the differential coefficients for independent a, b, c 's.

Analogous equations of course hold for the reverse transition from x, y, z to a, b, c as the independent variables. The functional determinant that then occurs:

$$D' = \begin{vmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & \frac{\partial a}{\partial z} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} & \frac{\partial b}{\partial z} \\ \frac{\partial c}{\partial x} & \frac{\partial c}{\partial y} & \frac{\partial c}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial a}{\partial x} & \frac{\partial b}{\partial x} & \frac{\partial c}{\partial x} \\ \frac{\partial a}{\partial y} & \frac{\partial b}{\partial y} & \frac{\partial c}{\partial y} \\ \frac{\partial a}{\partial z} & \frac{\partial b}{\partial z} & \frac{\partial c}{\partial z} \end{vmatrix} \quad . \quad . \quad (54)$$

denotes the ratio in which the volume of an element of the body changes when a material point is transferred from the position x, y, z to the position a, b, c. Since the two transitions cancel each other, we have:

$$D \, . \, D' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

This relationship also comes out directly if we apply the multiplication theorem (§ 5) of determinants, in its second form, to (51) and (54), and if we note that:

$$\left.\begin{aligned} \frac{\partial x}{\partial a} \cdot \frac{\partial a}{\partial x} + \frac{\partial x}{\partial b} \cdot \frac{\partial b}{\partial x} + \frac{\partial x}{\partial c} \cdot \frac{\partial c}{\partial x} = \frac{\partial x}{\partial x} = 1 \\ \frac{\partial x}{\partial a} \cdot \frac{\partial a}{\partial y} + \frac{\partial x}{\partial b} \cdot \frac{\partial b}{\partial y} + \frac{\partial x}{\partial c} \cdot \frac{\partial c}{\partial y} = \frac{\partial x}{\partial y} = 0, \text{ and so forth} \end{aligned}\right\} \quad (56)$$

Particular importance attaches not only to the volume dilatation $D - 1$ of an element of a body, but also to its rotation and its dilatation. Both are derived from the equations (32) by using the values (50) for the nine coefficients λ, μ, ν.

§ 12. We shall only carry out the calculation for the special case of *any arbitrary infinitely small transformation*, which we shall suppose characterized by having the displacement components (3) given as arbitrary infinitely small functions of a, b, c. All the theorems of § 9 then hold for an element of the body which contains the point a_0, b_0, c_0. In particular the rotation is given by (37) and

the dilatation by (38), the infinitely small coefficients (λ, μ, ν), by (50), (3) and (34), having the following values:

$$\left.\begin{aligned} \lambda &= \left(\frac{\partial u}{\partial a}\right)_0, & \lambda_2 &= \left(\frac{\partial u}{\partial b}\right)_0, & \lambda_3 &= \left(\frac{\partial u}{\partial c}\right)_0, \\ \mu_1 &= \left(\frac{\partial v}{\partial a}\right)_0, & \mu &= \left(\frac{\partial v}{\partial b}\right)_0, & \mu_3 &= \left(\frac{\partial v}{\partial c}\right)_0, \\ \nu_1 &= \left(\frac{\partial w}{\partial a}\right)_0, & \nu_2 &= \left(\frac{\partial w}{\partial b}\right)_0, & \nu &= \left(\frac{\partial w}{\partial c}\right)_0, \end{aligned}\right\} \quad . \quad (57)$$

By introducing the displacement components u, v, w, the letters x, y, z become superfluous in the significance they have hitherto had, since they can be replaced by $a + u, b + v, c + w$. Hence x, y, z are often used with the meaning that a, b, c previously had, and a, b, c are dispensed with entirely. But even if we retain the meaning of all the signs that have been used it is possible in the case of infinitely small displacements to replace the variables a, b, c in all the differential coefficients by x, y, z without introducing an appreciable error. For example, we have:

$$\frac{\partial u}{\partial a} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial a} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial a} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial a}$$

or by (3):

$$= \frac{\partial u}{\partial x} \cdot \left(1 + \frac{\partial u}{\partial a}\right) + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial a} + \frac{\partial u}{\partial z} \cdot \frac{\partial w}{\partial a} = \frac{\partial u}{\partial x} \quad . \quad (58)$$

except for small quantities of a higher order. Corresponding results hold for all other differential coefficients.

If we now again omit the suffix 0 everywhere, the components of the rotation are, by (37) and (57):

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \ \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \ \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad (59)$$

To calculate the dilatation we introduce the abbreviations:

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} = x_x, \ \frac{\partial v}{\partial y} = y_y, \ \frac{\partial w}{\partial z} = z_z \\ &\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = y_z, \ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = z_x, \ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = x_y \end{aligned}\right\} \quad . \quad (60)$$

where, obviously:

$$y_z = z_y, \; z_x = x_z, \; x_y = y_x \quad . \quad . \quad . \quad (60a)$$

By (42) and (57) the volume-dilatation is then:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = x_x + y_y + z_z \quad . \quad . \quad . \quad (61)$$

Further, by (57), the principal dilatations are the three roots of l in the cubic equation (41):

$$\begin{vmatrix} x_x - l & \frac{1}{2}x_y & \frac{1}{2}x_z \\ \frac{1}{2}y_x & y_y - l & \frac{1}{2}y_z \\ \frac{1}{2}z_x & \frac{1}{2}z_y & z_z - l \end{vmatrix} = 0 \quad . \quad . \quad . \quad (62)$$

The directions of the corresponding axes of dilatation are then obtained, by (40) and (57), from two of the equations:

$$\left.\begin{aligned} (x_x - l)\alpha + \frac{x_y}{2}\beta + \frac{x_z}{2}\gamma &= 0 \\ \frac{y_x}{2}\alpha + (y_y - l)\beta + \frac{y_z}{2}\gamma &= 0 \\ \frac{z_x}{2}\alpha + \frac{z_y}{2}\beta + (z_z - l)\gamma &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (63)$$

where the common sign of the direction cosines remains indeterminate.

Conversely, if the principal dilatations l, m, n and the corresponding axes of dilatation are given by their directions (α, β, γ), the components of the dilatation follow uniquely, by (38):

$$\left.\begin{aligned} x_x &= l\alpha_1^2 + m\alpha_2^2 + n\alpha_3^2, \; \cdots \\ \frac{1}{2}z_y = \frac{1}{2}y_z &= l\beta_1\gamma_1 + m\beta_2\gamma_2 + n\beta_3\gamma_3, \; \cdots \end{aligned}\right\} \quad . \quad (64)$$

§ 13. In vector calculus all these relationships, which, besides playing a characteristic part in the theory of deformable bodies, also figure prominently with a different

interpretation in several other branches of physics, are given special names and are designated by brief symbols. Thus the vector which results from the displacement vector $\boldsymbol{q}$, whose components are u, v, w, in such a way that the bracketed quantities of the equations (59) represent its components is called the "rotation" or "curl" of the vector $\boldsymbol{q}$, and to denote the rotation vector $\boldsymbol{o}$, whose components are ξ, η, ζ, we write instead of the three equations (59) the one equation:

$$\boldsymbol{o} = \frac{1}{2} \operatorname{curl} \boldsymbol{q} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (65)$$

The formal weakness which lies in the mechanical rotation being equal not to the whole, but to half the curl or rotor in (65), must be accepted now that this terminology has established itself; it makes itself a little less keenly felt because the operation denoted by curl does not find application in mechanics, but only in electrodynamics where there is no question of rotations.

Further, the scalar quantity which results from the displacement $\boldsymbol{q}$ through the operation indicated in (61) is called its "divergence" (div), and the volume dilatation (61) is written in the form:

$$\operatorname{div} \boldsymbol{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (66)$$

All reference to a definite co-ordinate system is also excluded in this case.

Lastly, concerning the dilatation in three mutually perpendicular directions, this is not characterized, like a vector, by a single directed quantity, but represents a higher configuration which is called a "tensor," in particular a tensor triplet of the second rank or second order, if we regard vectors as tensors of the first order. This tensor is determined by six mutually independent quantities—namely, either by the three "principal values" l, m, n in conjunction with the three corresponding mutually perpendicular "principal axes" ($\alpha_1, \ldots \gamma_3$), the two opposite directions of an axis being fully equiva-

lent and interchangeable, or by the six "components" x_x, y_y, z_z, $\frac{1}{2}y_z$, $\frac{1}{2}z_x$, $\frac{1}{2}x_y$. On account of the relations (60a), the tensor is called symmetrical. The relationships (62), (63) and (64) hold between the principal values, the directions of the principal axes and the components. In particular, if the principal values l, m, n are equal to each other (uniform dilatation), we have:

$$x_x = y_y = z_z = l = m = n$$
$$x_y = y_z = z_x = 0$$

and the principal axes are fully indeterminate.

Other properties of a symmetrical tensor will be obtained later (§ 20).

CHAPTER II

DYNAMICAL LAWS

§ 14. HAVING completed the purely kinematic part of our survey, we now turn to the dynamical part—that is, to the investigation of the forces which produce the deformation of a body, and we first inquire into the conditions of *equilibrium.*

Let us consider any body which is in a deformed state owing to the action of certain forces—for example, a bent rod, a twisted wire or a compressed gas. We shall divide the forces that act on the body into two classes :

1. Forces which act on all parts of the body, even the parts in the interior—"body-forces." We set these as, for example, gravity, proportional to the elements of mass of the body.

If $d\tau = dx \, . \, dy \, . \, dz$ denotes the volume, k the density of a mass-element, let the components of the body-force which acts on the mass-element be (cf. I, § 31) :

$$Xkd\tau, \; Ykd\tau, \; Zkd\tau \quad . \quad . \quad . \quad . \quad (67)$$

We regard the quantities X, Y, Z as finite.

2. Forces which act at the surface of the body—"surface-forces." For any portion σ of the surface of the body the quotient of the resultant force that acts on it and the area of surface of σ is called the "mean pressure" on σ. If the portion of surface shrinks up to a surface-element $d\sigma$, this quotient is simply called the "stress" on $d\sigma$, and is regarded as a finite quantity. To designate the surface-force that acts on $d\sigma$ we shall, since not only the position but also the direction is characteristic for $d\sigma$, append as a suffix the normal ν of $d\sigma$, directed towards the interior of the body, to the letters

X, Y, Z for the components. Then the components of the surface-force that acts on the surface-element $d\sigma$ are:

$$X_\nu d\sigma, \; Y_\nu d\sigma, \; Z_\nu d\sigma \quad . \quad . \quad . \quad . \quad (68)$$

The direction of this surface-force can form any arbitrary angle with the normal ν. If the two directions coincide, the surface-force acts in the sense of a compression of the body, like the pressure on a gas. If the directions are opposite, the surface-force acts in the sense of an expansion, like a tension on a stretched wire. If the directions are perpendicular to each other, the surface-force acts in the sense of a shear (§ 10), as in the case of torsion or friction.

Pressure, the resultant of the three pressure components X_ν, Y_ν, Z_ν, has, of course, the dimensions of the quotient of a force and a surface, or $[ML^{-1}T^{-2}]$ (cf. I, § 9).

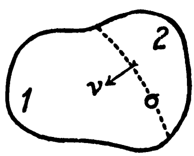

Fig. 1.

§ 15. The laws of equilibrium of a deformed body are all contained in the following law of general mechanics (I, § 112): if a system of points is in equilibrium, the external forces acting on the system, considered as rigid, are in equilibrium.

The wide applicability of this law is due to the circumstance that we can use any arbitrary part of the body as a system of points. For example, if we imagine the body to be divided into two parts 1 and 2 by any fictitious surface σ (Fig. 1), and choose the part 1 as the point system, then this part of the body, imagined as rigid, is in equilibrium under the action of the external forces that act on it. But these external forces also include, by I, § 112, besides the body-forces and the forces that act on the real surface, also the forces that act on the fictitious surface σ, which are due to the part 2 of the body and which are defined by having to be specially applied when the part 2 of the body is entirely removed, if the equilibrium is to remain undisturbed.

Now, according to the nomenclature we have chosen, every element $d\sigma$ of the surface of the part 1 of the body is acted on by a surface-force whose components are given by (68). In this way we are led to assume pressures which act in a perfectly definite sense even at every point in the interior of a body. But besides depending on the position, their magnitude also depends on the direction of the surface-element $d\sigma$ or of its normal ν, respectively. If we reverse the direction of ν—that is, if we regard the part 2 of the body as a point system—then (68) becomes replaced by the surface-force which the part 1 of the body exerts on the part 2 of the body in the surface-element $d\sigma$, and by the principle of the equality of action and reaction this force is exactly opposite to the former. Hence we have generally:

$$X_{-\nu} = -X_\nu,\ Y_{-\nu} = -Y_\nu,\ Z_{-\nu} = -Z_\nu\,. \quad . \quad (69)$$

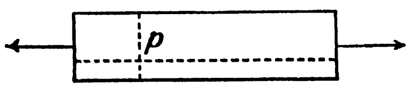

FIG. 2.

In order to see that the pressure at a definite point can assume very different values according to the direction of $d\sigma$, let us consider the example of a horizontal cylindrical rod, which is stretched in the direction of its length by opposite and equal forces (Fig. 2). Let us now suppose a horizontal surface to be drawn in its interior, and let us regard the portion of the rod that lies above it as a point system; then the pressure zero acts on a surface-element at P, because the equilibrium is not disturbed at all if we remove the lower part of the rod entirely. But if we draw a vertical plane through the same point P and regard the part of the rod to the left of it as a point system, then a more or less considerable pressure acts on a surface-element at P, since, if the right-hand part of the rod were to be removed, the equilibrium of the point system would be disturbed unless a particular force were applied.

The general law according to which the magnitude and direction of the pressure at a definite point P of a body depend on the direction of the normal of the surface-element drawn through P will be obtained in § 17 from the dynamical relationships.

§ 16. Now that the laws for the equilibrium of a deformed body have been reduced to those of general mechanics, we shall do the same for the more general case of the *motion* of a deformed body. This is accomplished simply by applying d'Alembert's Principle, according to which in any motion of a system of material points the external forces and the inertial resistances which act on a body which is supposed rigid maintain equilibrium at every moment (I, § 130).

Since the inertial resistance of a mass-element, namely:

$$-\frac{d^2x}{dt^2}k\,d\tau, \quad -\frac{d^2y}{dt^2}k\,d\tau, \quad -\frac{d^2z}{dt^2}k\,d\tau \quad . \quad . \quad (70)$$

is proportional to the mass of the element, it belongs to the body-forces, and the whole difference between the dynamics of a deformed body and the statics consists simply in the fact that the negative components of the acceleration become added to the components X, Y, Z of the forces (67) referred to unit mass.

Hence if we now apply the six equations of condition (I, (306*a*)) for the equilibrium of a rigid body to the present case and use the nomenclature that has been introduced, we get for the motion of a deformed body the following six equations:

$$\int\left(X - \frac{d^2x}{dt^2}\right)k\,d\tau + \int X_\nu\,d\sigma = 0, \text{ and so forth} \quad . \quad . \quad (71)$$

$$\int\left\{y\left(Z - \frac{d^2z}{dt^2}\right) - z\left(Y - \frac{d^2y}{dt^2}\right)\right\}k\,d\tau + \int(yZ_\nu - zY_\nu)\,d\sigma = 0$$
$$\text{and so forth} \quad . \quad . \quad . \quad . \quad . \quad (72)$$

where $d\tau$ denotes a volume-element and $d\sigma$ a surface-element of any arbitrarily chosen part of the body and

the integrations refer to this part. The equations, of course, also hold for the whole body.

The equations (71) and (72) represent completely the relationship between the forces and the accelerations. The following sections merely investigate its content a little further.

§ 17. We first apply equations (71) to an infinitely small element of the body situated somewhere in the interior, whose form is chosen as follows. Through any arbitrary point P we draw the three parallels to the positive directions of the co-ordinate axes (Fig. 3) and then intersect those three straight lines at a distance infinitely near P by a plane, shown in front of P in the figure. In this way we get a tetrahedron, at the vertex P of which the three edges meet at right angles. If we call the area of the face opposite P, $d\sigma$, its normal directed towards the interior of the tetrahedron ν, and the areas of the other three sides $d\sigma_x$, $d\sigma_y$, $d\sigma_z$ according to their normals, then the last-mentioned three faces represent the projections of $d\sigma$ on the three co-ordinate planes, and so we have the relationships :

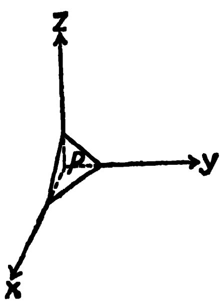

FIG. 3.

$$d\sigma_x = -d\sigma\cos(\nu x),\quad d\sigma_y = -d\sigma\cos(\nu y),$$
$$d\sigma_z = -d\sigma\cos(\nu z)\,. \quad . \quad . \quad . \quad (73)$$

since all the surface areas are positive, whereas the normal ν forms obtuse angles with all the co-ordinate directions.

In equation (71) the volume integral reduces in our case to a single term which contains the volume $d\tau$ of the tetrahedron as one of its factors, whereas the surface integral is equal to the sum of four terms, each of which corresponds to a face of the tetrahedron and is proportional to its area. Since $d\tau$ is of the third order of small quantities, whereas the $d\sigma$'s are of the second order of small quantities, the volume integral vanishes in com-

parison with each individual term of the surface integral, and the equation (71) becomes:

$$X_x d\sigma_x + X_y d\sigma_y + X_z d\sigma_z + X_\nu d\sigma = 0$$

in which we have taken into consideration that the inward normals of the faces of the tetrahedron are represented by x, y, z, ν. This, combined with (73), gives:

$$X_\nu = X_x \cos(\nu x) + X_y \cos(\nu y) + X_z \cos(\nu z)$$

and in the same way from the other two equations of (71): . . . (74)

$$Y_\nu = Y_x \cos(\nu x) + Y_y \cos(\nu y) + Y_z \cos(\nu z)$$
$$Z_\nu = Z_x \cos(\nu x) + Z_y \cos(\nu y) + Z_z \cos(\nu z)$$

These equations give for every point P of the body the relationship between the magnitude and direction of the pressures X_ν, Y_ν, Z_ν which act there and the direction of the normal ν of the surface-element on which the pressure acts (cf. end of § 15). For it makes no difference to the finite values of the pressure components whether $d\sigma$ passes exactly through the point P or is situated infinitely near it.

Accordingly, the pressure is determined for any arbitrary direction of ν so soon as the nine pressure components $X_x, \ldots Z_z$ are known. If ν coincides with a co-ordinate direction the equations are satisfied identically. They also satisfy generally the conditions (69), since the cosines reverse their sign when ν is replaced in the reverse direction.

Of the nine pressure components which correspond to the co-ordinate faces $d\sigma_x$, $d\sigma_y$, $d\sigma_z$, those that lie in the diagonal, X_x, Y_y, Z_z, represent the pressures which act normal to their surface; moreover, a positive value always denotes a compression as in a gas and is independent of the co-ordinate direction chosen, whereas a negative value always denotes a tension as in the case of a stretched wire (cf. § 14). For when the co-ordinate direction is reversed, both the component of the surface-force and

the inward normal change their signs. Of course three normal pressures need not all have the same sign.

The other six pressure components are the tangential pressures or shearing pressures (shears) such as occur in torsion and friction. Finally, we must bear in mind that all the above reflections refer to an infinitely small element of a body and that the values of the pressure components vary from element to element. Hence in general the nine pressure components X_x, . . . Z_z are to be regarded as functions of the point x, y, z.

§ 18. In order to be able to use the dynamical equations (71) and (72) for drawing further inferences, we must first

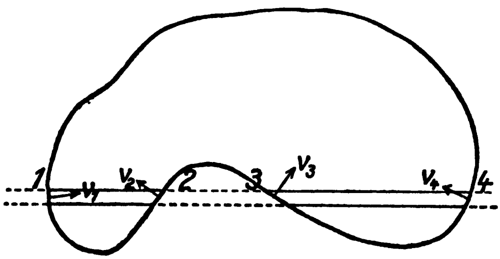

FIG. 4.

deduce a mathematical theorem which will also be of good service in other connections : it concerns the transformation of a certain space integral into a surface integral. Let ϕ denote a uniform continuous function of the space co-ordinates x, y, z. We proceed to find the value of the integral :

$$\int \frac{\partial \phi}{\partial x} \cdot d\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

taken over a definite region of space which, for the sake of generality, is drawn in Fig. 4 in such a way that its surface, when viewed from the outside, also has concavities. The surface may, indeed, be made up of quite different and separate pieces without the validity of the following theorems being impaired.

We set $d\tau = dx \,.\, dy \,.\, dz$ and first integrate with respect

to x, keeping y and z, and also dy and dz, constant—that is, we sum up over the volume-elements of an infinitely thin cylinder parallel to the x-axis and of cross-section dy, dz.

If the cylinder cuts the surface of the integration space at several points, 1, 2, 3, 4 it will be divided into several component cylinders, and the integral over the whole cylinder is equal to the sum of the integrals taken over the component cylinders—that is, in our case :

$$dy\,dz(\phi_2 - \phi_1 + \phi_4 - \phi_3) \quad . \quad . \quad . \quad (76)$$

If we denote the sizes of the surface-elements which the cylinder cuts out of the surface of the integration space by $d\sigma_1$, $d\sigma_2$, $d\sigma_3$, $d\sigma_4$, and their inward normals by ν_1, ν_2, ν_3, ν_4, then :

$$\begin{aligned} dydz &= d\sigma_1 \cos(\nu_1 x) = -d\sigma_2 \cos(\nu_2 x) = d\sigma_3 \cos(\nu_3 x) \\ &= -d\sigma_4 \cos(\nu_4 x) \quad . \quad . \quad . \quad . \quad (77) \end{aligned}$$

and the integral (76) taken over the cylinder is :

$$\begin{aligned} -\phi_1 \cos(\nu_1 x)\,d\sigma_1 - \phi_2 \cos(\nu_2 x)\,d\sigma_2 - \phi_3 \cos(\nu_3 x)\,d\sigma_3 \\ - \phi_4 \cos(\nu_4 x)\,d\sigma_4 \end{aligned}$$

A corresponding expression holds for each of the infinite number of infinitely thin cylinders parallel to the x-axis, into which the whole integration space can be imagined to be divided, and the sum of all these expressions, or the required integral, may be written :

$$\int \frac{\partial \phi}{\partial x} \cdot d\tau = -\int \phi \cos(\nu x)\,d\sigma \quad . \quad . \quad . \quad (78)$$

which is to be taken over all the elements $d\sigma$ of the surface of the integration space.

For example, if we set ϕ = const., we obtain the identity :

$$\int \cos(\nu x)\,d\sigma = 0$$

It is also easy to see directly that this identity is valid for any arbitrary form of the surface if we make the

sense of the equation clear to ourselves by using the relationship (77).

We shall here note several more general and more special applications of the formula just derived, which will be useful for later occasions.

If we denote any other uniform continuous function by ψ the equation:

$$\frac{\partial(\phi \,.\, \psi)}{\partial x} = \phi \frac{\partial \psi}{\partial x} + \psi \frac{\partial \phi}{\partial x}$$

when integrated over any space gives, according to (78), a surface integral on the left-hand side, and two space integrals on the right-hand side, so that we may write:

$$\int \phi \frac{\partial \psi}{\partial x} d\tau = -\int \phi\psi \cos(\nu x)\, d\sigma - \int \psi \frac{\partial \phi}{\partial x} d\tau \quad . \quad (79)$$

This relationship expresses the theorem of integration by parts applied to a region of three-fold manifold of dimensions.

The following is an application of equation (79) which is often of use:

$$\int \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial z} \right) d\tau =$$

$$- \int \left(\frac{\partial \phi}{\partial x} \cos(\nu x) + \frac{\partial \phi}{\partial y} \cos(\nu y) + \frac{\partial \phi}{\partial z} \cos(\nu z) \right) \psi d\sigma$$

$$- \int \psi \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) d\tau$$

or, by I, (120*a*) and I, (129):

$$\int \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial z} \right) d\tau = -\int \psi \frac{\partial \phi}{\partial \nu} d\sigma - \int \psi \,.\, \Delta\phi \,.\, d\tau \quad (80)$$

For $\psi = \phi$ we get, more particularly:

$$\int \left\{ \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 \right\} d\tau = -\int \phi \frac{\partial \phi}{\partial \nu} d\sigma - \int \phi \,.\, \Delta\phi \,.\, d\tau \quad (81)$$

and for $\psi = \text{const.}$:

$$\int \Delta\phi \,.\, d\tau = -\int \frac{\partial \phi}{\partial \nu} d\sigma \quad . \quad . \quad . \quad . \quad (82)$$

§ 19. Let us now again take, say, the first of the dynamical equations (71), applied to any arbitrary part of the body, and replace the X_ν which occurs in it by (74); we may then write the surface integral by (78) as a space integral, since, for example:

$$\int X_x \cos(\nu, x) d\sigma = -\int \frac{\partial X_x}{\partial x} d\tau$$

and may combine this space integral with the first integral of (71) to form a single space integral. If we then choose the part of the body as infinitely small, of volume $d\tau$, then the space integral reduces to a single term, and we get, omitting the factor $d\tau$:

$$\left(X - \frac{d^2x}{dt^2}\right)k - \frac{\partial X_x}{\partial x} - \frac{\partial X_y}{\partial y} - \frac{\partial X_z}{\partial z} = 0$$

and in the same way:

$$\left(Y - \frac{d^2y}{dt^2}\right)k - \frac{\partial Y_x}{\partial x} - \frac{\partial Y_y}{\partial y} - \frac{\partial Y_z}{\partial z} = 0 \qquad (83)$$

and:

$$\left(Z - \frac{d^2z}{dt^2}\right)k - \frac{\partial Z_x}{\partial x} - \frac{\partial Z_y}{\partial y} - \frac{\partial Z_z}{\partial z} = 0$$

which are valid for every point of the body.

The characteristic feature of these equations is that in them the absolute values of the pressure components do not occur but only their space differential coefficients.

A uniform pressure, no matter how great, can never produce a motion; to bring about movement a space variation of pressure is required or a "pressure gradient." In this respect the physical meaning of a pressure resembles that of a potential (cf. I, § 39, *et seq.*).

In the dynamical equations (72) we can in the same way transform the surface integral into a space integral after the following model:

$$\int y Z_x \cos(\nu x) d\sigma = -\int \frac{\partial(yZ_x)}{\partial x} d\tau$$

and so obtain similarly from the first of the equations (72):

$$\left\{y\left(Z - \frac{d^2z}{dt^2}\right) - z\left(Y - \frac{d^2y}{dt^2}\right)\right\} k$$
$$- \frac{\partial}{\partial x}(yZ_x - zY_x) - \frac{\partial}{\partial y}(yZ_y - zY_y) - \frac{\partial}{\partial z}(yZ_z - zY_z) = 0$$

or, if we eliminate the body-forces by means of (83) and perform the differentiations:

$$Z_y = Y_z. \text{ In the same way, } X_z = Z_x \text{ and } Y_x = X_y \quad (84)$$

Thus the tangential components of the pressure are equal in pairs, and the nine pressure components X_x, . . . Z_z become reduced quite generally to six.

The equations (83) and (84) are fully equivalent physically to the equations (71) and (72), since we can pass from them by integrating over any part of the body to the latter. In the sequel we shall base all our investigations into problems of equilibrium and motion on these equations.

§ 20. Now that the relationships (84) have enabled us to simplify considerably the laws (74) which govern the pressure conditions at a definite point of the body we shall next examine the form of these laws more closely. We begin by inquiring whether, if the six pressure components X_x, . . . Z_z are arbitrarily given, there are surface-elements on which the pressure acts perpendicularly—that is, for which the direction of the pressure coincides with the direction of the normal at the point. The condition for this is:

$$X_\nu = p\cos(\nu x),\quad Y_\nu = p\cos(\nu y),\quad Z_\nu = p\cos(\nu z)$$

where p denotes the value of the pressure. Substituted in (74) this gives:

$$\left.\begin{aligned}(X_x - p)\cos(\nu x) + X_y\cos(\nu y) + X_z\cos(\nu z) &= 0\\ Y_x\cos(\nu x) + (Y_y - p)\cos(\nu y) + Y_z\cos(\nu z) &= 0\\ Z_x\cos(\nu x) + Z_y\cos(\nu y) + (Z_z - p)\cos(\nu z) &= 0\end{aligned}\right\} \quad (85)$$

and these equations, except for the nomenclature and

the fact that we are here dealing with only finite quantities, are fully identical with the equations (40) or (63), and they also lead to the same results, which we shall epitomize in the following statements.

The pressure state in an element of a body is always represented by a *symmetrical tensor*, the pressure or tension tensor, whose components constitute the six pressure components and whose principal values are the so-called "principal pressures" p, q, r, the latter being the roots of a cubic equation of the form (62). The principal axes of the tensor are the normals of those surface-elements on which the pressure acts perpendicularly; so they are simultaneously the directions of the corresponding principal pressures. The whole pressure state is just as much determined by the values and the directions of the principal pressures by equations of the form of (64) as by the six pressure components with regard to any co-ordinate system. If, in particular, the three principal pressures are equal to one another (as is always the case in a completely elastic liquid, § 44), the normal pressures $X_x = Y_y = Z_z = p = q = r$, the tangential pressures $X_y = Y_z = Z_x = 0$, and the directions of the principal axes are indeterminate.

In order to form a picture of the general case of the relationship between the pressure (X_ν, Y_ν, Z_ν) and the direction of the normal ν of the surface-element, we imagine the following ideal surface of the second order (ellipsoid or hyperboloid) to be constructed:

$$X_x x^2 + Y_y y^2 + Z_z z^2 + 2Y_z yz + 2Z_x zx + 2X_y xy = \pm 1 \quad (86)$$

where the sign on the right-hand side is to be chosen in such a way that the geometric operations that are to be performed are real.

The following theorem may be proved about this surface: the direction of the surface normal at the end of a diameter whose direction is ν gives the direction of the pressure which acts on a surface-element which is perpendicular to ν.

For if $f(x, y, z) = 0$ denotes the equation to the surface, then:

$$\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

are the direction ratios of the normals at the point (x, y, z) of the surface, and if this point lies on the diameter whose direction is ν, then:

$$x : y : z = \cos(\nu x) : \cos(\nu y) : \cos(\nu z)$$

If these values are used in conjunction with the expression (86) for the function f the ratios (87) become changed into $X_\nu : Y_\nu : Z_\nu$, as a glance at (74) shows.

According to the physical meaning which we have just attributed to the surface (86) its form and position are independent of the choice of co-ordinate system. Its axes are obviously the axes of the principal pressures: for $p = q = r$ it is a sphere.

In general, it may be asserted that in passing from the co-ordinates x, y, z to any other orthogonal rectilinear co-ordinates x', y', z' the expression (86) remains invariant. For the equation which we obtain instead of (86) if we write x', y', z' in place of x, y, z and at the same time $X'_x, \ldots Z'_z$, instead of $X_x, \ldots Z_z$, represents the same surface. The relationship between the accented pressure components $X'_{x'}, \ldots$ and the unaccented $X_x, \ldots$ must therefore be such that on replacing the accented components and co-ordinates by the unaccented co-ordinates and components the equation (86) again results. This theorem contains the laws of transformation of the pressure components in passing from one co-ordinate system to any other.

It is understood that all the relationships here derived are correspondingly valid not only for the pressure tensor, but equally for the deformation tensor and, indeed, for every symmetrical tensor of the second order. Only its physical significance in the different cases is quite different and more or less important. The moment of inertia (I, § 142) of a body with respect to the different

directions that pass through a definite point is also represented by a symmetrical tensor, one whose principal values (principal moments of inertia) are always positive. The surface (86) then constitutes the ellipsoid of inertia. We are here, as so often, led to make the interesting observation that Nature, although she represents her phenomena to us externally in an enormous variety of ways, nevertheless often uses the same simple means in regions very remote from one another. If this were not so, it would be infinitely more difficult for the human mind, which operates mostly with similes and deductions from analogies, to elucidate her laws.

Before we proceed to deal with fruitful applications of the dynamical equations (83) we must become acquainted with the relationships which exist between pressure and deformation, and the question of these relationships necessarily leads us to consider the *material* of which the body in question is composed, which we have hitherto left entirely out of account. This will therefore have to engage our attention in the sections which immediately follow.

PART TWO

INFINITELY SMALL DEFORMATIONS

CHAPTER I

RIGID BODIES. GENERAL REMARKS

§ 21. As we have seen in the first part of this volume, we still require to know, in order to formulate unambiguously the laws of motion of deformable bodies, only the relationship between the deformation tensor and the pressure tensor. In order to establish this relationship we introduce, above all, the hypothesis—which for the present we shall everywhere retain in the sequel—that the pressure always and everywhere depends solely and alone on the simultaneous local deformation and conversely. This assumption is by no means always fulfilled in Nature; for, strictly speaking, in the case of all rigid bodies their deformations exhibit to a greater or less degree a dependence not only on the momentary forces, but also on the nature of the treatment which they have previously undergone, and further also on the temperature, which can in general change quite independently of pressure and deformation. When and so far as the hypothesis above introduced actually applies to a body, we call this body "perfectly elastic."

It is also, of course, a condition for perfect elasticity that a body which has been subjected for a time to arbitrary deforming forces should assume, when the forces have ceased to act, no other state of equilibrium than that which it had before the forces were applied; for the deformation zero must correspond uniquely to the pressure zero. Thus if a body exhibits the phenomenon of elastic hysteresis or fatigue—that is, if, when the deforming forces have ceased to act, it assumes its original state of equilibrium only gradually—it does *not*

behave perfectly elastically. On the other hand, the magnitude of the deformation caused by a definite pressure plays no part in the question of perfect elasticity. In this respect scientific terminology deviates a little from the usage of ordinary speech, since in everyday life we associate the idea of great elasticity also with that of particularly great deformability, and in this sense we call rubber, for example, more elastic than glass. In the scientific sense the reverse is the case: glass is more elastic than rubber in so far as it exhibits the property of elastic hysteresis to a far less marked degree than rubber.

Experiment shows that every body may be regarded as perfectly elastic so long as its deformation does not exceed a certain value, which is called the "limit for perfect elasticity." Hence our investigations in the sequel refer to all *bodies*, but only to such *deformations* as are so small that they lie within the elastic limit. To simplify the mathematical treatment we assume the deformations to be infinitely small.

§ 22. A body in which all directions are physically of equal value is called "isotropic"; on the other hand, a body in which some directions exhibit different physical behaviour are called "anisotropic." For isotropic bodies we can give a relationship between deformation and pressure at once; for since the pressure is completely determined by the deformation, in the case of any arbitrary deformation the principal axes of the pressure tensor must coincide with those of the deformation tensor. In the case of anisotropic bodies, however, this conclusion cannot be drawn, because here certain favoured directions of the body will yet play a part. Since our next discussion is also to include crystals—that is, anisotropic bodies—we must follow a more general line of treatment if we wish to make progress.

The deformation tensor is characterized by its six components x_x, x_y, . . . (§ 12), the pressure tensor by its six components X_x, X_y, . . . (§ 19); and the latter quantities are definite functions of the former. Now

since the deformation components become infinitely small, we need retain only the first terms in the expansion in Taylor's series, and we obtain the result that the pressure components are *linear* functions of the deformation components and, as we may at once add, *homogeneous* functions, since we wish to reckon the deformations from the state which corresponds to the pressure zero—that is, from the "natural" state of the body. The pressure components then vanish simultaneously with the deformation components.

Thus if we attach the indices 1, 2, 3, 4, 5, 6 to the deformation components in the order of sequence x_x, y_y, z_z, y_z, z_x, x_y and the same indices to the pressure components in the order of sequence X_x, Y_y, Z_z, Y_z, Z_x, X_y, then we obtain as the general expression for the dependence of the pressure on the deformation the six equations :

$$\left.\begin{aligned} X_x &= a_{11}x_x + a_{12}y_y + a_{13}z_z + a_{14}y_z + a_{15}z_x + a_{16}x_y \\ Y_y &= a_{21}x_x + a_{22}y_y + a_{23}z_z + a_{24}y_z + a_{25}z_x + a_{26}x_y \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned}\right\} \quad (88)$$

in which the thirty-six constants a are determined by the material constitution of the body. If these expressions are substituted in the equations of motion (83), we get the general laws of motion.

The next question is whether the constants a are entirely independent of one another, or whether there are definite relationships between them which when introduced into the equations (88) simplify them. It is quite clear that if there are any symmetries in the structure of a crystal the case will be specialized and simplified in that the number of constants a will become less than 36. But for the present we shall still deal with the general case of a completely unsymmetrical crystal, in order that we may subject it to a condition which we have not introduced hitherto, but which we know always to be fulfilled in Nature—namely, the law of conservation of energy. We shall see that the introduction of this law

leads to a general and considerable simplification of the system of equations (88).

§ 23. According to the principle of conservation of energy I, (393), the change that occurs in the total energy $L + U$ of a material system in the element of time dt is equal to the work performed by the external forces on the system in the same time:

$$d(L + U) = A \quad . \quad . \quad . \quad . \quad . \quad (89)$$

We shall now apply this equation to the present case by selecting any arbitrary part of the whole body, exactly as in § 15, and using it as the material system on which to found our discussion.

As for the first part of the energy, the kinetic energy L, it is simply the sum of the kinetic energies of all the mass-elements $k \,.\, d\tau$ of the part of the body under consideration—that is:

$$L = \frac{1}{2}\int\left\{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2 + \left(\frac{dw}{dt}\right)^2\right\} \cdot kd\tau \quad . \quad (90)$$

where u, v, w denote the components of displacement of a material point from its natural position, and the integration is to be taken over the part of the body in question.

We also know of the second part of the energy, the potential energy U, that it is likewise composed of the potential energies of the individual mass-elements; so it has the form:

$$U = \int F \,.\, d\tau \quad . \quad . \quad . \quad . \quad . \quad (91)$$

where the function F, the potential energy per unit volume, depends, on account of the perfect elasticity of the body, only on the state of deformation of the volume-element in question—that is, on the six deformation components x_x, y_y, ... Since in the case of potential energy we are concerned only with its changes in value, and not with its absolute value, we shall without loss of generality set $F = 0$ for the natural state of the body, which is its state when free of deformations.

Concerning lastly the external work A, this consists,

by § 14, of the work of the body-forces which act on all the elements of the part of the body in question and, besides, on the work of the pressures which act from without on the points of the surface of the same part of the body—that is:

$$A = \int (Xdu + Ydv + Zdw)kd\tau + \int (X_\nu du + Y_\nu dv + Z_\nu dw)d\sigma \quad (92)$$

where the first integral is to be taken over the volume, the second over the surface of the part of the body.

To apply the postulate (89) of the energy principle we first form the expression for dL. Since the mass $kd\tau$ does not depend on the time, we may also differentiate (90) directly:

$$dL = \int \left(\frac{d^2u}{dt^2} du + \frac{d^2v}{dt^2} dv + \frac{d^2w}{dt^2} dw \right) kd\tau$$

or, if we substitute for the three components of acceleration their values from the equations of motion (83):

$$\begin{aligned} dL = & \int (Xdu + Ydv + Zdw)kd\tau \\ & - \int d\tau \, . \left\{ \begin{aligned} & du \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} \right) \\ + & dv \left(\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} \right) \\ + & dw \left(\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} \right) \end{aligned} \right\} \quad . \quad . \quad . \quad (93) \end{aligned}$$

If we now make use of the transformation formula (79) by applying it to each of the nine space integrals into which the second space integral resolves, we obtain, if we simultaneously take (74) into account:

$$\begin{aligned} dL = & \int (Xdu + Ydv + Zdw)kd\tau \\ & + \int d\tau \, . \left\{ \begin{aligned} & X_x \frac{\partial du}{\partial x} + X_y \frac{\partial du}{\partial y} + X_z \frac{\partial du}{\partial z} \\ + & Y_x \frac{\partial dv}{\partial x} + Y_y \frac{\partial dv}{\partial y} + Y_z \frac{\partial dv}{\partial z} \\ + & Z_x \frac{\partial dw}{\partial x} + Z_y \frac{\partial dw}{\partial y} + Z_z \frac{\partial dw}{\partial z} \end{aligned} \right\} \quad . \quad . \quad (94) \\ & + \int d\sigma \, . \, (X_\nu du + Y_\nu dv + Z_\nu dw) \end{aligned}$$

This expression for dL, when substituted in the energy equation (89), simplifies it to a considerable degree in that the term on the right-hand side which is due to the external work A, and which is represented by (92), cancels out with the equal term on the left-hand side. The only terms left in the energy equation are then the potential energy dU and the second space integral of (94). The latter can be written more simply. In the first place, we must bear in mind that in the expressions $\frac{\partial du}{\partial x}$ and so forth the sign ∂ refers to the space differentiation, but the sign d to the time differentiation, so that these two operations are quite independent of each other. Hence we have:

$$\frac{\partial du}{\partial x} = d\frac{\partial u}{\partial x}, \frac{\partial du}{\partial y} = d\frac{\partial u}{\partial y}, \text{ and so forth.}$$

If, further, we use the relationships (84) and the abbreviations (60), the energy equation finally assumes the form:

$$dU + \int(X_x dx_x + Y_y dy_y + Z_z dz_z + Y_z dy_z + Z_x dz_x + X_y dx_y)d\tau = 0 \quad . \quad . \quad (95)$$

By (91) we may now write for the change in the potential energy:

$$dU = \int dF \, . \, d\tau \quad . \quad . \quad . \quad . \quad . \quad (96)$$

Strictly speaking, we should add another term here which is due to the time change in the value of the volume-element $d\tau$—namely, the term $\int F d(d\tau)$, where the first d refers to the time change, the second d to the space change. But this term may be neglected, since in the case of infinitely small deformations the time changes of a volume-element, even for finite times, are infinitely small compared with the size of the volume-element, whereas the time changes of the potential energy F are

of the same order of magnitude as the amount of this energy.

By using (96) we may write the expression (95) as a single space integral which is to be taken over the volume of the part of the body in question. Now if we assume the part of the body to be infinitely small—that is, a single volume-element $d\tau$, the integral sign may be omitted, and we obtain for every individual element of the body:

$$-dF = X_x dx_x + Y_y dy_y + Z_z dz_z + Y_z dy_z + Z_x dz_x + X_y dx_y$$

But since F is determined by the six deformation components, we have:

$$dF = \frac{\partial F}{\partial x_x} dx_x + \frac{\partial F}{\partial y_y} dy_y + \frac{\partial F}{\partial z_z} dz_z + \frac{\partial F}{\partial y_z} dy_z + \frac{\partial F}{\partial z_x} dz_x + \frac{\partial F}{\partial x_y} dx_y$$

and since the deformation components and their changes are independent of one another, we have in general:

$$\left.\begin{aligned} X_x &= -\frac{\partial F}{\partial x_x}, \quad Y_y = -\frac{\partial F}{\partial y_y}, \quad Z_z = -\frac{\partial F}{\partial z_z} \\ Y_z &= -\frac{\partial F}{\partial y_z}, \quad Z_x = -\frac{\partial F}{\partial z_x}, \quad X_y = -\frac{\partial F}{\partial x_y} \end{aligned}\right\} \quad . \qquad (97)$$

That is, *the components of the pressure tensor are the negative derivatives of a single function of the deformation components with respect to these components*, just as in the case of central forces the force components are the negative derivatives of a single function of the co-ordinates, the potential, with respect to those co-ordinates (I, (107)). Hence the potential energy F of unit volume is also called the *elastic potential*.

By (88) the elastic potential is a quadratic function of the deformation components and, moreover, it is a homogeneous function, since not only the linear but, by the assumption we made above, also the absolute term vanishes.

§ 24. From the results that have been obtained we see

that the general expression of the elastic potential has the form:

$$
\begin{aligned}
F = \frac{a_{11}}{2} x_x^2 &+ a_{12} x_x y_y + a_{13} x_x z_z + a_{14} x_x y_z + a_{15} x_x z_x + a_{16} x_x x_y \\
&+ \frac{a_{22}}{2} y_y^2 + a_{23} y_y z_z + a_{24} y_y y_z + a_{25} y_y z_x + a_{26} y_y x_y \\
&+ \frac{a_{33}}{2} z_z^2 + a_{34} z_z y_z + a_{35} z_z z_x + a_{36} z_z x_y \\
&+ \frac{a_{44}}{2} y_z^2 + a_{45} y_z z_x + a_{46} y_z x_y \\
&+ \frac{a_{55}}{2} z_x^2 + a_{56} z_x x_y \\
&+ \frac{a_{66}}{2} x_y^2
\end{aligned} \tag{98}
$$

From this we obtain by (97) the six pressure components as linear homogeneous functions of the deformation components as above in (88), but with the difference that now $a_{12} = a_{21}$, and so forth. In other words, the existence of an elastic potential amounts to the same thing as the condition that in the constants of elasticity a_{ij} the two indices i and j may be interchanged. As compared with the general theory developed in § 22, we have therefore effected the considerable simplification that the elastic behaviour of a body depends no longer on thirty-six, but only on twenty-one constants. We are able at this stage to make only one definite assertion about these constants, without entering into the special properties of the body—namely, that they fulfil the condition of making the potential F positive in all circumstances. For the elastic potential is the potential energy of the deformation, and this has the property that when the body passes from the deformed state back to its natural state—that is, when the potential goes from the value F to the value zero—it transforms into kinetic, that is positive, energy; and this requires that it itself should be positive. This is an example of the application of the general theorem that, corresponding to a stable

state of equilibrium, there is always a minimum of the potential energy (I, § 105). Now since the natural (undeformed) state of the body represents a stable state of equilibrium, the potential energy in every state of deformation is greater than that in the natural state—that is, greater than zero.

From this we further conclude that the six constants $a_{11}, a_{22}, \ldots a_{66}$ are all positive. For if, for example, a_{11} were negative, we should only need to make x_x differ from zero, while assuming all the other deformation components to be zero, to obtain a negative potential. But the remaining constants a must also satisfy certain irregularities, which we shall not, however, formulate here.

§ 25. In practice we usually require to find the change effected in a body by given external forces. The first important question which we encounter is that regarding the *uniqueness* of the solution—that is, whether there is, corresponding to given external pressures, a perfectly definite change in the body, or whether several different changes are compatible with the equations of the problem. We shall restrict ourselves to considering the case of equilibrium.

We therefore now assume the pressures X_ν, Y_ν, Z_ν that act on the surface of the body, and also the body-forces X, Y, Z as given; and we assume that three functions u, v, w of x, y, z have been found which, regarded as displacements of the points x, y, z of the body, satisfy all the conditions of equilibrium—namely, both the equations (83) for the interior:

$$\left.\begin{aligned} Xk &= \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} \\ Yk &= \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} \\ Zk &= \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_x}{\partial z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (99)$$

and the equations (74) for the surface. In all these equations we must insert for the pressure components $X_x, \ldots$ those values which are obtained by calculation

from the expressions for the displacements u, v, w, when we first form, by (60), the deformation components that correspond to them, and then, by (97), the pressure components.

We shall now further assume that three other functions u', v', w' of x, y, z exist, which differ from the previous functions, but which also satisfy all the conditions of equilibrium. Then the same equations (99) and (74) also hold for those pressure components X'_x, X'_y, . . ., as are obtained if we calculate the deformation components x'_x, x'_y, . . ., by (60), from u', v', w' instead of from u, v, w, and substitute these values in (97).

Let us now consider the functions:

$$u_0 = u' - u, \; v_0 = v' - v, \; w_0 = w' - w \quad . \quad . \quad (100)$$

of x, y, z and investigate that change of the body which they, regarded as displacement components, represent. In the first place, we get from them, by (60), the deformation components:

$$x_{x_0} = x'_x - x_x, \; x_{y_0} = x'_y - x_y, \; . \; . \; . \quad . \quad . \quad (101)$$

and further, by (97), the pressure components:

$$X_{x_0} = X'_x - X_x, \; X_{y_0} = X'_y - X_y, \; . \; . \; . \quad . \quad (102)$$

On the other hand, if we subtract the equations (99) for the unaccented pressure components from the same equations (99) for the accented pressure components and if we bear in mind that the given body-forces X, Y, Z are the same in both systems of equations, it follows that:

$$0 = \frac{\partial X_{x_0}}{\partial x} + \frac{\partial X_{y_0}}{\partial y} + \frac{\partial X_{z_0}}{\partial z}, \; . \; . \; . \quad . \quad . \quad (103)$$

and, finally, for the surface of the body, by subtracting the unaccented from the accented equations (74), since the given surface pressures X_ν, Y_ν, Z_ν are the same in each case:

$$0 = X_{x_0} \cos(\nu x) + X_{y_0} \cos(\nu y) + X_{z_0} \cos(\nu z), \; . \; . \; . \quad (104)$$

A doubt might arise here as to whether in these subtractions the values of the direction cosines in both

systems of equations may be regarded as the same, since the form of the surface of the body will be different in the two changes. We can easily convince ourselves that the error incurred through this is of a smaller order of magnitude. For the direction cosines of the normal of a surface-element are finite; so their values for any infinitely small change of the body deviate only by an infinitely small amount from those values which they have in the natural state of the body, whereas the pressure components $X_x, \ldots$ vary in these changes by amounts which are considerable compared with the values of the pressure components themselves.

The equations (103) and (104) have a very clear physical meaning. For they express the conditions for a change u_0, v_0, w_0, in which external forces act neither on the elements of mass nor on the surface of the body. This reduces the question, which was raised above, as to the uniqueness of the solution for u, v, w in the case where the external forces are given to the simpler question whether in the absence of any external influence a change u_0, v_0, w_0 of the body is possible which differs from zero.

We can find the answer by means of the following reasoning. If we multiply the three equations (103) in turn by u_0, v_0, w_0, add them together, multiply them by the volume-element $d\tau$, and finally integrate over the whole body, we obtain an equation with nine space integrals, each of which may be transformed in the following way by (79):

$$\int u_0 \frac{\partial X_{x_0}}{\partial x} d\tau = -\int \frac{\partial u_0}{\partial x} \cdot X_{x_0} . d\tau - \int u_0 X_{x_0} \cos(\nu x) d\sigma.$$

This transforms the equation, in view of (104), into:

$$\int (X_{x_0} x_{x_0} + Y_{y_0} y_{y_0} + Z_{z_0} z_{z_0} + Y_{z_0} y_{z_0} + Z_{x_0} z_{x_0} + X_{y_0} x_{y_0}) d\tau = 0$$

or, if we substitute for the pressure components their values given by (97):

$$\int F_0 . d\tau = 0$$

where F_0 denotes the expression (98) if we attach the suffix zero to all the deformation components in it. Now F_0 is always positive, and is zero only in the limiting case where the deformation energy vanishes. Hence it follows from the last equation that F_0 vanishes for every individual volume-element, and hence that all the deformation components $x_{x_0}, x_{y_0}, \ldots$ are individually equal to zero. In other words, if no external forces of any kind act on the body the deformation is necessarily equal to zero.

The next question is whether the change u_0, v_0, w_0 also then vanishes. To decide this we must investigate whether, when the six deformation components vanish:

$$\left.\begin{aligned} \frac{\partial u_0}{\partial x} &= 0, \quad \frac{\partial v_0}{\partial z} + \frac{\partial w_0}{\partial y} = 0 \\ \frac{\partial v_0}{\partial y} &= 0, \quad \frac{\partial w_0}{\partial x} + \frac{\partial u_0}{\partial z} = 0 \\ \frac{\partial w_0}{\partial z} &= 0, \quad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (105)$$

the displacement components u_0, v_0, w_0 also necessarily become equal to zero. This is by no means the case, as may easily be seen. Rather, u_0, v_0, w_0, the general solutions of the differential equations (105) are obtained in the following way. For u_0 the following three conditions hold:

$$\frac{\partial u_0}{\partial x} = 0, \quad \frac{\partial^2 u_0}{\partial y^2} = 0, \quad \frac{\partial^2 u_0}{\partial z^2} = 0$$

of which the last two result by differentiating the equations which contain $\frac{\partial u_0}{\partial y}$ and $\frac{\partial u_0}{\partial z}$. Thus u_0 is of the form:

and likewise:

$$\left.\begin{aligned} u_0 &= \lambda + \lambda_2 y + \lambda_3 z + \lambda' yz \\ v_0 &= \mu + \mu_1 x + \mu_3 x + \mu' xz \\ w_0 &= \nu + \nu_1 x + \nu_2 y + \nu' xy \end{aligned}\right\} \quad . \quad . \quad (106)$$

where the twelve coefficients λ, μ, ν are constant. Further, by (105):

$$\begin{aligned} \mu_3 + \mu' x + \nu_2 + \nu' x &= 0 \\ \nu_1 + \nu' y + \lambda_3 + \lambda' y &= 0 \\ \lambda_2 + \lambda' z + \mu_1 + \mu' z &= 0 \end{aligned}$$

Hence:

$$\lambda' = 0, \qquad \mu' = 0, \qquad \nu' = 0$$
$$\nu_2 + \mu_3 = 0,\ \lambda_3 + \nu_1 = 0,\ \mu_1 + \lambda_2 = 0.$$

These six conditions make the expressions (106) completely identical with the expressions (39) for the most general infinitely small change which a body can undergo without experiencing a deformation—that is, for a translation and a rotation—a result which is immediately evident and might have been predicted. We may therefore now enunciate the following theorem: the external forces that act in the interior and on the surface of the body do not, indeed, uniquely determine the displacement components u, v, w, but the changes represented by the various solutions of the problem differ from each other only by translations and rotations of the whole body. Thus they all correspond to the same deformation, and in this sense we may say that the problem of equilibrium is uniquely solved by the equations that have been set up. Later, when dealing with similar problems, we shall choose the six constants which remain undetermined quite arbitrarily, since they are of no further interest to us.

§ 26. **Bodies of Symmetrical Structure. Crystal Systems.** The twenty-one constants on which the elastic behaviour of a homogeneous crystal depends may be conveniently studied by arranging them in the following scheme:

$$\left.\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{matrix}\right\} \quad . \quad . \quad (107)$$

But the values of the individual constants a do not depend on the nature of the crystal alone, but also in general on the choice of the co-ordinate system—namely on the orientation of the co-ordinate axes with respect to the directions which play a favoured part in the structure of the crystal. This follows immediately if we reflect that the value of the elastic potential F—that is, the elastic energy—cannot possibly depend on the choice of co-ordinates. Hence if we transform any deformations x_x, x_y, . . . to another co-ordinate system x', y', z', and if we denote the new deformation components by adding an accent, we get by (98) that :

$$\frac{a_{11}}{2} x_x^2 + a_{12} x_x y_y + \ldots \equiv \frac{a'_{11}}{2} x_x'^2 + a'_{12} x'_x y'_y + \ldots \qquad (108)$$

and from this identity, if the accented deformation components are expressed in terms of the unaccented components, we find, by equating the individual corresponding terms on both sides, the relationships between the accented and the unaccented constants which correspond to the relationships between the accented and the unaccented co-ordinates. In general, the accented constants a' will entirely or partly differ from the unaccented constants a.

Now if the crystal has the special property that for a definite change of the co-ordinate system all the a''s are equal to all the a's—that is, if the constants a are invariant with respect to the change of co-ordinates—we say that a definite kind of *symmetry* is peculiar to the crystal. If there are several changes of co-ordinates of this kind, the symmetry of the crystal is of higher order.

We may also express every kind of condition of symmetry, instead of by the invariance of the elastic constants a with respect to the corresponding co-ordinate transformation, in another way—namely, by retaining the original co-ordinate system, but subjecting the crystal to such a change that finally it is orientated towards the original co-ordinate system just as it was orientated,

according to the first point of view, when it was at rest, towards the accented co-ordinate system. The presence of the symmetry is then shown by the fact that after the change the crystal behaves in all directions exactly as before the change, or that, as we say, the crystal is made to coincide with itself as a result of the change. These two kinds of definition of symmetry are obviously completely equivalent and are only different forms of the same thing. For our purpose the first formulation, the condition of invariance of a with respect to a transformation of co-ordinates, is more convenient. We shall therefore make use of it in the sequel.

Crystals are divided into various classes according to the degree of symmetry they show. We see from the above remarks that the principle of this classification is not fixed from the outset and cannot be uniquely determined by direct inference, but is to a certain extent arbitrary and dictated only by questions of expediency. We shall here make use of the division into six systems of crystals which has long been in use, and for this purpose we most conveniently apply as the transformation of co-ordinates a rotation of the co-ordinate system.

If the elastic constants a of a crystal are all invariant with respect to a rotation of the co-ordinate system about a co-ordinate axis of angle $\frac{2\pi}{n}$ (or, if the crystal is made to coincide with itself by means of a rotation about the angle $\frac{2\pi}{n}$) we call this axis an "n-fold axis of symmetry" of the crystal.

Of course, an axis of symmetry is not a definite straight line, but only a definite direction, since all parallel straight lines are fully equivalent.

The existence of a one-fold axis of symmetry is not a real condition of symmetry at all, since a rotation through the angle 2π does not alter the co-ordinate system at all. Crystals which have only one-fold axes of symmetry form the last (the sixth) or *asymmetrical*

(*triclinic*) system. Copper sulphate is an example of them. Its elastic behaviour depends on twenty-one constants, as is represented in the scheme (107).

If a crystal has no higher symmetry than a single two-fold axis of symmetry, it belongs to the fifth or *monosymmetrical* (*monoclinic*) system, as, for example, mica or sodium bicarbonate. To find the conditions for the elastic behaviour of crystals of this system, we first inquire generally into the relationship between the accented and the unaccented elastic constants a when the co-ordinate system is rotated through the angle $\frac{2\pi}{2} = \pi$ about the z-axis. We obtain for this :

$$x' = -x, \ y' = -y, \ z' = z$$

and correspondingly :

$$u' = -u, \ v' = -v, \ w' = w$$

and by (60) :

$$x'_x = x_x, \ y'_y = y_y, \ z'_z = z_z$$
$$y'_y = -y_z, \ z'_x = -z_x, \ x'_y = x_y$$

If we substitute these values in the identity (108), it follows that of the twenty-one constants a' thirteen are equal to the a's that bear the same suffixes, whereas for the others we have :

$$a'_{14} = -a_{14}, \ a'_{15} = -a_{15}, \ a'_{24} = -a_{24}, \ a'_{25} = -a_{25}$$
$$a'_{34} = -a_{34}, \ a'_{35} = -a_{35}, \ a'_{46} = -a_{46}, \ a'_{56} = -a_{56}$$

These relationships hold quite generally for every crystal. If the z-axis is a two-fold axis of symmetry all the quantities a are invariant, and for this reason all those of them which change their sign in the transformation must vanish. Hence for the elastic behaviour of a monosymmetrical crystal we have, by (107), the scheme :

$$\left.\begin{matrix} a_{11} & a_{12} & a_{13} & 0 & 0 & a_{16} \\ & a_{22} & a_{23} & 0 & 0 & a_{26} \\ & & a_{33} & 0 & 0 & a_{36} \\ & & & a_{44} & a_{45} & 0 \\ & & & & a_{55} & 0 \\ & & & & & a_{66} \end{matrix}\right\} \quad . \quad . \quad (109)$$

If besides the z-axis one of the other co-ordinate axes—say, the x-axis—is also a two-fold axis of symmetry, we arrive at the fourth or *rhombic* system, which counts among its representatives saltpetre, aragonite and topaz. The following scheme corresponds to it, which can be found in the same way as the above :

$$\left.\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ & a_{22} & a_{23} & 0 & 0 & 0 \\ & & a_{33} & 0 & 0 & 0 \\ & & & a_{44} & 0 & 0 \\ & & & & a_{55} & 0 \\ & & & & & a_{66} \end{array}\right\} \quad . \quad . \quad . \quad (110)$$

Its structure informs us immediately that in this case the third co-ordinate axis—that is, the y-axis—is also a two-fold axis of symmetry.

We arrive from the rhombic system at the third or *tetragonal* (*quadratic*) system (for example, zirconium) if we introduce the further condition that one of the co-ordinate-axes—say, the z-axis—is a four-fold axis of symmetry. For a rotation through the angle $\frac{\pi}{2}$ about the z-axis the following transformation formulæ hold :

$$\begin{array}{lll} x' = y, & y' = -x, & z' = z \\ u' = v, & v' = -u, & w' = w \\ x'_x = y_y, & y'_y = x_x, & z'_z = z_z \\ y'_z = -x_z, & z'_x = z_y, & x'_y = -x_y. \end{array}$$

By (108) this gives for the nine constants of the rhombic system the general relationships :

$$\begin{array}{lll} a'_{11} = a_{22}, & a'_{12} = a_{12}, & a'_{13} = a_{23}, \\ a'_{22} = a_{11}, & a'_{23} = a_{13}, & a'_{33} = a_{33}, \\ a'_{44} = a_{55}, & a'_{55} = a_{44}, & a'_{66} = a_{66} \end{array}$$

Now if all the accented a''s are to be equal to the corresponding unaccented a's, it follows that :

$$a_{11} = a_{22}, \quad a_{13} = a_{23}, \quad a_{44} = a_{55}$$

and hence, by (98), the elastic potential of a tetragonal crystal is:

$$F = \frac{a_{11}}{2}(x_x^2 + y_y^2) + a_{12}x_xy_y + a_{13}(x_x + y_y)z_z$$
$$+ \frac{a_{33}}{2}z_z^2 + \frac{a_{44}}{2}(y_z^2 + z_x^2) + \frac{a_{66}}{2}x_y^2 \quad . \quad . \quad (111)$$

The existence of a three-fold or a six-fold axis of symmetry is the condition for the second or *hexagonal* system, which is represented by numerous crystals such as nitre, calcspar, graphite, tourmaline and ice.

Lastly, the first or regular system emerges from the tetragonal system if also a second co-ordinate axis, and consequently also the third axis, is a four-fold axis of symmetry. The expression for the elastic potential of a regular crystal comes out, by (111) as:

$$F = \frac{a_{11}}{2}(x_x^2 + y_y^2 + z_z^2) + a_{12}(x_xy_y + y_yz_z + z_zx_x)$$
$$+ \frac{a_{44}}{2}(y_z^2 + z_x^2 + x_y^2) \quad . \quad . \quad (112)$$

—that is, it still depends on three constants. The regular system includes rock salt, fluorspar, diamond.

§ 27. **Isotropic Bodies.** The necessary and sufficient condition for a body to be elastically isotropic—that is, that it should have no favoured directions at all—is that its elastic constants should all be invariant with respect to *any* change of the co-ordinate system, or, what amounts to the same thing, that it can be made to coincide with itself by means of any arbitrary rotation. In order to deduce the conditions consequent upon this for the values of the elastic constants, we find it most simple to refer the deformation to the principal dilatations and the directions of the axes of the principal dilatations—that is, we reduce the six deformation components x_x, y_y, . . . by means of the equations (64) to terms of the three principal dilatations l, m, n and the nine direction cosines of the axes of the principal dilatations α_1, . . . γ_3, and substitute these values in the expression (112) for the

elastic potential. In an isotropic body the elastic potential must be entirely determined by the quantities l, m, n, which we may also take as the direction cosines, and it obviously forms a symmetrical homogeneous quadratic function of l, m, n.

The meaning of this postulate can be most easily grasped if we direct our attention to all the symmetrical homogeneous quadratic functions of l, m, n. There are only two of them which are independent of each other, namely:

$$l^2 + m^2 + n^2 \quad \text{and} \quad lm + mn + nl$$

All others can be reduced to terms of these two. For example:

$$(l + m + n)^2 = (l^2 + m^2 + n^2) + 2(lm + mn + nl) \quad (113)$$

Hence it follows that the elastic potential of an isotropic body has only two constants which are independent of each other, and is, say, of the form:

$$F = \frac{\lambda}{2}(l + m + n)^2 + \mu(l^2 + m^2 + n^2) \quad . \quad (114)$$

where λ and μ are positive.

To express F in terms of the deformation constants x_x, x_y, . . . we reflect that, since l, m, n are the roots of the cubic equation (62):

$$l + m + n = x_x + y_y + z_z \quad . \quad . \quad . \quad (115)$$

since this is the coefficient of l^2, and:

$$lm + mn + nl = (y_y z_z + z_z x_x + x_x y_y) - \tfrac{1}{4}(y_z{}^2 + z_x{}^2 + x_y{}^2) \quad (116)$$

since this is the coefficient of l in the equation. Hence it follows by (113) that:

$$l^2 + m^2 + n^2 = x_x{}^2 + y_y{}^2 + z_z{}^2 + \tfrac{1}{2}(y_z{}^2 + z_x{}^2 + x_y{}^2)$$

(which may also be confirmed directly by (64)), and by (114):

$$F = \frac{\lambda}{2}(x_x + y_y + z_z)^2 + \mu\left(x_x{}^2 + y_y{}^2 + z_z{}^2 + \frac{y_z{}^2 + z_x{}^2 + x_y{}^2}{2}\right) \quad (117)$$

If we use the following abbreviation for the volume dilatation:

$$x_x + y_y + z_z = \sigma \quad . \quad . \quad . \quad . \quad (118)$$

we get the pressure components for an elastic isotropic body, by (97), expressed in terms of the deformation components in the following way:

$$\left.\begin{aligned} X_x &= -\lambda\sigma - 2\mu x_x, & Y_z &= -\mu y_z, \\ Y_y &= -\lambda\sigma - 2\mu y_y, & Z_x &= -\mu z_x, \\ Z_z &= -\lambda\sigma - 2\mu z_z, & X_y &= -\mu x_y. \end{aligned}\right\} \quad . \quad (119)$$

Thus these equations form the supplement, which we stated to be necessary at the beginning of the first chapter, § 21, to the general equations of motion for an elastic isotropic body.

CHAPTER II

STATES OF EQUILIBRIUM OF RIGID BODIES

§ 28. We shall now make some applications of the theory that has been developed to statical problems involving rigid bodies, and shall restrict ourselves for simplicity to isotropic bodies. Since body-forces, like gravity, usually play only a very subordinate part in the deformations of rigid bodies, we shall leave them out of consideration here. The conditions of equilibrium (99) which hold in the interior then simplify to :

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0, \quad . \quad . \quad . \quad \quad . \quad . \quad (120)$$

whereas the conditions for the surface of the body run, unchanged, as in (74) :

$$X_\nu = X_x \cos(\nu x) + X_y \cos(\nu y) + X_z \cos(\nu z), \quad . \quad . \quad . \quad . \quad (121)$$

Every displacement u, v, w, whose deformation components x_x, x_y, . . ., as calculated from (60), lead, by (119), to expressions for the pressure components X_x, X_y, . . . which satisfy the equations (120), represents a possible state of equilibrium in Nature; the deformation involved corresponds to those forces which act on the surface of the body which are represented by the pressure components X_ν, Y_ν, Z_ν in (121).

Very great importance attaches here to the theorem proved in § 25 that the deformation is *uniquely* determined by the external forces. So when we have found a solution of the conditions of equilibrium, no matter by what method, we may be sure that it is the only solution which corresponds to the external forces in

question. For in Nature the problem usually presents itself in the form: given the external forces, what are the deformations produced by them? On account of the complicated structure of the equations, it is quite impossible to solve the problem by first writing down the general solution of the differential equations (120) and then calculating the undetermined constants in them from the boundary conditions (121). Rather, it is usually better to start out from the boundary conditions (121) and then endeavour by trial to find a particular solution of (120) which is adequate to those boundary conditions. If the endeavour is successful, "the" solution of the problem has been found. We shall use this method in the sequel as a rule.

§ 29. Let us first consider a compression of a body of any arbitrary form which is uniform in all directions; this is the so-called *cubical compression.* A given uniform pressure p then acts everywhere on the surface in the normal direction. Hence we have:

$$X_\nu = p\cos(\nu x),\ \ Y_\nu = p\cos(\nu y),\ \ Z_\nu = p\cos(\nu z) \quad . \quad (122)$$

They are the given components of external pressure. The boundary conditions (121) are obviously satisfied if we assume that everywhere on the surface:

$$\left.\begin{aligned} X_x = Y_y = Z_z = p \\ X_y = Y_z = Z_x = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (123)$$

We shall further assume tentatively that these six equations represent the values of the pressure components not only at the surface but also in the whole interior. The equations (120) are then satisfied. Moreover, we obtain from (119):

$$p = -\lambda\sigma - 2\mu x_x, \ . \ . \ . \ y_z = 0, \ . \ . \ . \quad . \quad (124)$$

from which, by adding up the first three equations, we get:

$$3p = -3\lambda\sigma - 2\mu(x_x + y_y + z_z)$$

and by (118):

$$\sigma = -\frac{3p}{3\lambda + 2\mu}, \quad x_x = y_y = z_z = \frac{\sigma}{3} \quad . \quad (125)$$

Hence if we set:

$$u = \frac{\sigma}{3}x, \quad v = \frac{\sigma}{3}y, \quad w = \frac{\sigma}{3}z \quad . \quad . \quad . \quad (126)$$

then, by (60), the equations (124) are satisfied, and we have obtained the solution of the equation. The generalization which can be obtained by taking up certain other constants into the expressions for u, v, w does not, by § 25, produce any change in the deformation that has been found.

So a general uniform pressure of amount p produces a general uniform contraction of volume of amount (125), which is independent of the form of the body. Hence the constant $\frac{3}{3\lambda + 2\mu}$ is also called the "cubical compressibility" of the substance, and its reciprocal value:

$$\frac{p}{-\sigma} = \lambda + \frac{2}{3}\mu \quad . \quad . \quad . \quad . \quad (127)$$

the "modulus of cubical elasticity."

§ 30. Having considered cubical elasticity, we next turn our attention to *linear elasticity*—that is, we investigate the equilibrium of a body which is stretched in one direction—say, that of the x-axis—and, to avoid the calculation becoming too involved, we shall assume the body to be in the form of a cylinder which is parallel to the x-axis (wire, rod), of length l and of arbitrary cross-section. Let the front cross-section lie in the yz-plane—that is, $x = 0$, and be held fast there by constraining forces. Let the rear cross-section, $x = l$, be acted on by an external force of given value F in the direction of the positive x-axis, while no external forces at all are to act on the curved surface or mantle of the cylinder. We wish to inquire into the resulting deformation.

First, to satisfy the surface conditions (121) again we consider the curved surface of the cylinder. Here $X_\nu = Y_\nu = Z_\nu = 0$; further, $\cos(\nu x) = 0$, whereas $\cos(\nu y)$ and $\cos(\nu z)$ can assume arbitrary values. So the surface conditions are satisfied if we set:

$$X_y = Y_z = Z_x = 0, \quad Y_y = Z_z = 0 \quad . \quad . \quad (128)$$

We shall also, as a trial, assume these values to be valid in the whole interior, so that of the six pressure components only X_x differs from zero. The differential equation (120) is then actually satisfied if we assume besides that X_x is constant. The value of this constant is then obtained from the surface condition for the free cross-section $x = l$; for the condition for the fixed cross-section $x = 0$ gives us only the value of the constraining forces which hold the cross-section fast and which are of no further interest to us here.

At the free cross-section we have:

$$\cos(\nu x) = -1, \quad \cos(\nu y) = \cos(\nu z) = 0$$

and hence by (121) and (128):

$$X_\nu = -X_x, \quad Y_\nu = 0, \quad Z_\nu = 0.$$

Now, since X_x is constant, we get for the resultant of all the parallel pressures which act on the elements of the free cross-section:

$$\int X_\nu d\sigma = -X_x \, . \, q = F \quad . \quad . \quad . \quad (129)$$

where q denotes the value of the cross-section. Hence:

$$X_x = -\frac{F}{q} \quad . \quad . \quad . \quad . \quad . \quad (130)$$

From the pressure components we now also obtain the deformation components by (119) in the following way. First, we again have:

$$y_z = z_x = x_y = 0$$

Moreover, from:

$$\frac{F}{q} = \lambda\sigma + 2\mu x_x$$
$$0 = \lambda\sigma + 2\mu y_y$$
$$0 = \lambda\sigma + 2\mu z_z$$

we obtain by addition:

$$\frac{F}{q} = 3\lambda\sigma + 2\mu\sigma$$

and so:

$$\sigma = \frac{1}{3\lambda + 2\mu} \cdot \frac{F}{q} \quad . \quad . \quad . \quad . \quad . \quad (131)$$

from which:

$$x_x = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} \cdot \frac{F}{q}, \; y_y = z_z = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} \cdot \frac{F}{q} \quad . \quad (132)$$

With these values the displacements:

$$u = x_x \,.\, x, \; v = y_y \,.\, y, \; w = z_z \,.\, z \quad . \quad . \quad (133)$$

represent the complete solution of the problem.

As was to be expected, the deformation that has been found is associated with a volume dilatation; but the amount σ of this dilatation in the case of a linear tension is, as a comparison of the expressions (131) and (125) shows, only the third of that which occurs in the case of cubical tension. Concerning the dilatations of the individual axes, it is obvious that that of the x-axis is positive. Corresponding to it there is, by (133), a displacement of the free cross-section—that is, an extension of the cylinder by the amount:

$$u = x_x \,.\, l = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} \cdot \frac{lF}{q} \,. \quad . \quad . \quad (134)$$

That is, the extension is directly proportional to the tension and to the length and inversely proportional to the cross-section of the cylinder, and, lastly, directly proportional to a material constant whose reciprocal

value is also called the "modulus of linear elasticity" E of the substance :

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad . \quad . \quad . \quad . \quad . \qquad (135)$$

But by (132) the dilatation of the length of the cylinder is associated with a contraction of every direction perpendicular to the axis of the cylinder, whose value may be obtained most conveniently by writing down the ratio of the transverse contraction to the longitudinal dilatation :

$$\frac{-y_y}{x_x} = \frac{\lambda}{2(\lambda + \mu)} = \epsilon < \frac{1}{2} \cdot \quad . \quad . \quad . \qquad (136)$$

By measuring the two constants E and ϵ we may calculate λ and μ, and hence determine the whole elastic behaviour of the body.

The value of ϵ is very small for cork but very large, near the limiting value $\frac{1}{2}$, for rubber. For metals and glasses we can assume it to be equal to about $\frac{1}{3}$ to a first approximation, but considerable differences often manifest themselves in the individual substances.

The linear modulus of elasticity E, whose dimensions, by (135) and (119), are the same as those of a pressure, is of the following order of magnitude for metals and glasses :

$$E \sim 10^{12} \text{ dynes/cm.}^2 \text{ (cf. I, equation (8}a\text{))} \quad . \quad . \qquad (136a)$$

Hence the coefficients of elasticity λ and μ are of the same order of magnitude.

§ 31. We shall next consider a case of *torsion* (twisting), and shall investigate this for a cylinder of length l whose lower cross-section, the base of the cylinder, we shall assume to be held fixed in the xy-plane ($z = 0$), whereas the upper, free cross-section ($z = l$) is rotated in its own plane through a certain angle L by means of external forces. Let the mantle again be uninfluenced by external forces.

For the sake of variety we shall this time strike out in the reverse direction by beginning, not with the ex-

ternal forces, but with the assumed deformation and by inquiring into the forces that they produce. Let the deformation consist in every cross-section of the cylinder parallel to the xy-plane being rotated in its plane without distortion through an angle which is proportional to the distance z of the cross-section from the base-plane. Hence if ω denotes the angle of rotation of the horizontal base-plane, the angle of rotation of a cross-section at the height z is:

$$\frac{\omega \cdot z}{l}$$

From this we obtain uniquely the values of the displacement components u, v, w by introducing the cylindrical co-ordinates:

$$x = \rho \cos \phi, \; y = \rho \sin \phi, \; z = z \quad . \quad . \quad (136b)$$

After what has been said, we see that the material points x, y, z have the following co-ordinates after the deformation:

$$\begin{aligned} x + u &= \rho \cos\left(\phi + \frac{\omega z}{l}\right), \\ y + v &= \rho \sin\left(\phi + \frac{\omega z}{l}\right) \\ z + w &= z \end{aligned}$$

Consequently, in view of the fact that ω is infinitely small:

$$u = -\frac{\omega y z}{l}, \; v = \frac{\omega x z}{l}, \; w = 0 . \quad . \quad . \quad (137)$$

and by (60):

$$\left.\begin{aligned} x_x &= 0, \quad y_y = 0, \quad z_z = 0 \\ y_z &= \frac{\omega x}{l}, \quad z_x = -\frac{\omega y}{l}, \quad x_y = 0 \end{aligned}\right\} \quad . \quad . \quad (138)$$

By (119) these deformation components yield the pressure components:

$$\left.\begin{aligned} X_x &= 0, \quad Y_y = 0, \quad Z_z = 0 \\ Y_z &= -\frac{\mu \omega x}{l}, \quad Z_x = \frac{\mu \omega y}{l}, \quad X_y = 0 \end{aligned}\right\} . \quad . \quad (139)$$

Since these expressions satisfy the equations (120) for the interior identically, it follows that the assumed deformation represents a state of equilibrium which is possible in Nature and which must always occur when the pressures which are to be calculated from (121) act on the surface of the cylinder.

Let us first consider the pressures on the curved surface of the cylinder which vanish according to the assumption we have made. Since for this surface $\cos(\nu z) = 0$, whereas $\cos(\nu x)$ and $\cos(\nu y)$ have arbitrary values, it follows from (121), if we use (139) for all points on the mantle of the cylinder:

$$X_\nu = 0, \; Y_\nu = 0, \; Z_\nu = \frac{\mu\omega}{l}(y\cos(\nu x) - x\cos(\nu y)). \quad (140)$$

From this we see that the deformation assumed by us is compatible with the further assumption that no external forces act on the mantle of the cylinder only if at all points of the curved surface of the cylinder:

$$y\cos(\nu x) - x\cos(\nu y) = 0$$

This is a purely geometrical condition; it asserts that the direction $x : y$—that is, the direction of a radius vector which is drawn at right-angles to the z-axis from any point on it—coincides with the direction $\cos(\nu x) : \cos(\nu y)$—that is, with the direction of the normal ν at the extremity of the radius vector—or, in other words, that the cross-section is a circle, so that the cylinder is circular.

Hence if we wish to retain our assumptions we find ourselves compelled to restrict the further deductions that we make to the special case of the *circular cylinder*. With this restriction we then obtain for the external pressure that acts on an element of the upper extreme face of the cylinder, from (121) and (139), since here:

$$\cos(\nu x) = 0, \; \cos(\nu y) = 0, \; \cos(\nu z) = -1$$

the value:

$$X_\nu = -\frac{\mu\omega y}{l}, \; Y_\nu = \frac{\mu\omega x}{l}, \; Z_\nu = 0 \quad . \quad . \quad (141)$$

or, by (136*b*):

$$X_\nu = -\frac{\mu\omega\rho}{l}\sin\phi, \quad Y_\nu = \frac{\mu\omega\rho}{l}\cos\phi, \quad Z_\nu = 0$$

The external plane thus acts on every element of the upper free end-face of the cylinder in the plane of the face and perpendicularly to the direction of the radius vector ρ, in the sense of the angle of torsion ω, as is easy to understand. All pressures:

$$X_\nu d\sigma, \quad Y_\nu d\sigma, \quad Z_\nu d\sigma$$

which act on the elements of the surface give the following resultant, by I (306), since $d\sigma = \rho d\rho d\phi$:

$$\boldsymbol{F}_x = \int X_\nu d\sigma = 0, \quad \boldsymbol{F}_y = \int Y_\nu d\sigma = 0, \quad \boldsymbol{F}_z = \int Z_\nu d\sigma = 0$$

and a resultant couple:

$$\boldsymbol{N}_x = -\int zY_\nu d\sigma = 0, \quad \boldsymbol{N}_y = \int zX_\nu d\sigma = 0$$

$$\boldsymbol{N}_z = \int (xY_\nu - yX_\nu)d\sigma$$

$$\boldsymbol{N}_z = \frac{\mu\omega}{l}\iint \rho^3 d\rho d\phi = \frac{\pi}{2}\frac{\mu\omega}{l}\cdot r^4 = N \quad . \quad . \quad . \quad (142)$$

where r denotes the radius of the cylinder.

From this we deduce conversely that if the upper extreme face of a circular cylinder of which the base-plane is kept fixed is acted on by a couple which lies in its plane and which has the turning moment N, the cylinder will experience a torsion of the kind assumed, the upper extreme face being turned through the angle:

$$\omega = \frac{2}{\pi}\frac{lN}{\mu r^4} \quad . \quad . \quad . \quad . \quad . \quad (142a)$$

Thus the angle of torsion is proportional to the length of the cylinder and the turning moment of the couple, but inversely proportional to the fourth power of the radius and of the material constant μ, which is therefore also called the "modulus of torsion" of the substance.

§ 32. But what torsion does a cylinder undergo when it is not of circular, but, say, of elliptic cross-section and

when its upper end-face is acted on by the same couple N while the mantle is not acted on by external forces? We shall now discuss this related problem in some detail.

In the first place, it is clear that for this more general case we must modify the simple assumptions made at the beginning of the previous section, according to which every cross-section is turned in its plane without distortion; for these assumptions apply, as we saw, only to a circular cylinder. But it is easy to see in the following convincing way what sort of modification we must make. Although the simple deformation represented by the equations (137) for a cylinder of any arbitrary cross-section no longer applies, it yet represents a state of equilibrium that is possible in Nature, as we inferred from the fact that the equations (20) are satisfied for the interior of the cylinder—that is, this deformation necessarily occurs if the appropriate external forces act on the cylindrical surface. Which external pressures must then act on the curved surface of the cylinder can be seen immediately from the equations (140), in the last of which we need only substitute those values for the direction cosines of the inward normal which are conditioned by the form of the cylindrical cross-section. Accordingly the external pressure on the mantle is everywhere parallel to the z-axis—that is, it acts upwards or downwards according as Z_ν is positive or negative.

In general we may write:

$$Z_\nu = \frac{\mu\omega}{l} \cdot r \sin \delta \quad . \quad . \quad . \quad . \quad . \quad (143)$$

where δ denotes the angle which the outward normal at a point of the mantle makes with the radius vector $r = \sqrt{x^2 + y^2}$ which lies in the cross-section in question, when the outward normal appears turned about the z-axis in the positive sense with respect to the radius vector. For a circular cross-section we have, of course, that $\delta = 0$, but for an elliptic cross-section, whose major

and minor axes coincide with the x- and y-axes, respectively, δ and hence also Z_ν are positive in the first and third quadrants, when ω is positive, and negative in the second and fourth quadrants. Hence if an elliptic cylinder in the position in question is twisted, then, in order that the cross-sections may simply be rotated in their planes, external shearing forces must act on the mantle of the cylinder, upwards in the first and third quadrants, downwards in the second and fourth quadrants. From this it follows that if these external forces do *not* act, the points in question of the mantle experience a displacement in the *opposite* direction, which causes the originally plane surfaces to be indented, downwards in the first and third quadrants ($w<0$), upwards in the two others ($w>0$). Only those points of the mantle in which the length of the radius vector r becomes a maximum or minimum ($\delta = 0$) when we pass round the cross-section—that is, the vertices in the case of an ellipse—retain their level ($w = 0$); the others rise or fall.

To get a general idea of the sense of the distortion—one which is independent of the choice of co-ordinate system, of the position of the fixed base-plane, the sign of ω and the direction of the external turning moment which acts on the free end-plane—it is useful to consider the distinction between the different kinds of screw-lines. When a point describes a screw-line it executes a turn simultaneously with a forward movement which is perpendicular to the plane of the turn. Now if the screw-line is such that in describing the screw or, what comes to the same thing, in turning the screw through a fixed nut, the positive axis of the rotation (I, § 83) coincides with the direction of the forward motion, as, for example, in the case of an ordinary cork-screw, the screw-line is called a right-handed screw; in the opposite case, it is a left-handed screw. For this distinction it is immaterial in which direction the point describes the screw-line or the screw is turned. For when the sense of the

rotation is reversed, so is the direction of the forward motion and also that of the positive axis of rotation.

Reverting to our present case, let us consider a filament or thread of the cylinder which is originally parallel to the axis of torsion; then after the deformation has been effected the filament forms a screw-line. On the other hand, the boundary points of a cylindrical cross-section which was originally plane forms, when the deformation is complete, a wave-line. The theorem holds quite generally, then, that those parts of this wave-line which surround the minima of the radius vector are portions of screw-lines of the same kind as the filaments of the cylinder above considered; if the latter are right-handed screws, so are the former. For the parts of the wave-line which are adjacent to the maxima of r the reverse holds.

In the example of the elliptic cross-section considered above and in the nomenclature there adopted the filaments of the cylinder are right-handed screws. Consequently the boundary wave-line at the extremities of the semi-minor-axis also runs like a right-handed screw—that is, it rises in passing from the first to the second and from the third to the fourth quadrants, exactly as was established above.

To solve this problem quantitatively we must consider the analytical conditions. In dealing with the case of an elliptic cylinder whose semi-axes are a and b, the results hitherto obtained suggest to us to generalize the expressions (137) of the displacement components in the following way:

$$u = -\frac{\omega yz}{l}, \quad v = \frac{\omega xz}{l}, \quad w = -Cxy \quad . \quad . \quad (144)$$

where C denotes a positive constant. For this value of w exhibits the property found above of being negative in the first and third quadrants and positive in the second and fourth quadrants. Through this the conditions (120) for the interior and (121) for the curved surface

of the cylinder are actually fulfilled, the former identically and the latter by setting:

$$C = \frac{\omega}{l} \cdot \frac{a^2 - b^2}{a^2 + b^2} \quad . \quad . \quad . \quad . \quad . \quad (145)$$

This completes the solution of the torsion problem also for an elliptic cylinder. The expression (142) for the external turning moment N in the case of a circular cylinder here becomes generalized, by (121), to:

$$N = \frac{\pi\mu\omega}{l} \cdot \frac{a^3b^3}{a^2 + b^2} \quad . \quad . \quad . \quad . \quad . \quad (146)$$

CHAPTER III

VIBRATIONS IN RIGID BODIES

§ 33. If a motion is to be permanently associated with infinitely small deformations, which we have taken as our general assumption in this second half of the present volume, it is clear that the motion must not be in one direction only, but must alternate in different directions—that is, it must consist in a " vibration " of the body, in which the displacements, and with them the deformations, continually change their signs. We are thus here dealing with vibrations in elastic rigid bodies, which for simplicity we shall assume to be isotropic. The laws which govern these processes have already been established in the first chapter. They have been formulated in the equations of motion (83), which, if we disregard gravitational forces, may be written as follows :

$$\left.\begin{aligned} k\frac{\partial^2 u}{\partial t^2} &= -\frac{\partial X_x}{\partial x} - \frac{\partial X_y}{\partial y} - \frac{\partial X_z}{\partial z} \\ k\frac{\partial^2 v}{\partial t^2} &= -\frac{\partial Y_x}{\partial x} - \frac{\partial Y_y}{\partial y} - \frac{\partial Y_z}{\partial z} \\ k\frac{\partial^2 w}{\partial t^2} &= -\frac{\partial Z_x}{\partial x} - \frac{\partial Z_y}{\partial y} - \frac{\partial Z_z}{\partial z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (147)$$

as well as in the surface conditions (74) and in the relationships (119) between the pressure tensor and the deformation tensor for an isotropic substance.

By far the most interesting cases of such vibration processes relate to bodies whose dimensions are not of the same order of magnitude in all three dimensions of space, but which extend predominantly only in two dimensions or even one. This, of course, at once sim-

plifies the laws of motion considerably, since the number of independent space co-ordinates becomes less. On the other hand, if we wish to pay due regard to the most important vibration processes that occur in practice, it is advantageous to generalize somewhat the equations set up earlier. For in § 22 we calculated the displacement components u, v, w, and with them the deformation components from that state of the body in which all the external pressures are zero. This made the pressure components homogeneous functions of the deformation components, and in the undeformed state all the pressure components were equal to zero. But in Nature it is often just those vibrations in which this restriction is absent that are of interest. For example, if we take the vibrations of a string of a violin, the displacements u, v, w will have to be calculated from that state of the body in which it is in stable equilibrium. This is thus the state of no deformation. But the pressure components are by no means zero in it; rather, a certain and indeed comparatively strong tension exists in the string. In fact, this tension is a factor which influences the character of the vibrations so essentially that the influence of the elastic constants λ, μ which are characteristic of the material of the string is practically negligible in comparison with it. For example, a string of catgut would not vibrate at all without external tension. Here we may therefore speak in a certain sense of an artificial or forced elasticity which does not depend on the material, but only on the external forces. To take these conditions adequately into account we shall therefore generalize the relationships (119) between the pressure components and the deformation components by regarding the pressure components as linear functions of deformation components, as we have done heretofore, but no longer as homogeneous functions of them; and we shall adapt the absolute terms in these functions to the stable condition of equilibrium that happens to rule in the body.

For this purpose, as has already been emphasized in the above example, we introduce besides the material constants also a new constant into the equations of motion; and according as the influence of the first or the second constant predominates we obtain entirely different vibration phenomena—namely, those due to natural elasticity and those due to forced elasticity. These differences are so important in practice that they have also been kept completely distinct in everyday language. Of one-dimensional bodies, "rods" vibrate with natural elasticity, but "strings" with forced elasticity; of two-dimensional bodies "plates" and "bells" vibrate with natural, but membranes and tympani with forced elasticity. In acoustics, so far as solid bodies are concerned, the vibrations with forced elasticity are by far the more important. Hence we shall deal with these primarily and choose as the simplest case the vibrations of a string or wire which is so tightly stretched that the influence of the elasticity of the material, the so-called stiffness of the wire, is utterly negligible compared with that of the tension.

§ 34. **Tightly Stretched Wire or String.** Let the wire or string, which is to have the form of a cylinder with an infinitely small cross-section, coincide, in its position of rest, with the x-axis. Every point of the string is then characterized by a definite value of x; and the motion of the string is fully known if the displacement components u, v, w have been found as functions of x and t. To solve this problem we shall above all introduce a generalization of the relationships (119) which we pointed out as necessary in the preceding section. The equilibrium state of the string is identical with the equilibrium, discussed in § 30, of a cylinder stretched in one direction; so the equations (128) and (129) of § 30 also apply here, if F denotes the stretching force and q the cross-section of the string. Thus these equations give us the values of the pressure components for the undeformed state of the string. Hence the

desired generalization of (119) follows for our case uniquely as :

$$\left.\begin{aligned} X_x &= -\frac{F}{q} - \lambda\sigma - 2\mu x_x, & Y_z &= -\mu y_z \\ Y_y &= \qquad - \lambda\sigma - 2\mu y_y, & Z_x &= -\mu z_x \\ Z_z &= \qquad - \lambda\sigma - 2\mu z_z, & X_y &= -\mu x_y \end{aligned}\right\} \quad . \qquad (148)$$

The assumption that the string is "tightly" stretched expresses itself in the condition that the first term that occurs in the expression X_x and that is due to the tension F is great compared with each of the following terms. On the other hand, F must not be assumed to be arbitrarily great. For since σ and x_x are infinitely small, X_x and accordingly also $\frac{F}{q}$ are small compared with λ and μ. The order of magnitude of the tension F is thus enclosed within certain limits. If we consider, however, that, for example, for metals the coefficients λ and μ are, by (136*a*) of the order of magnitude 10^{12} dynes/cm.^{2}, which corresponds to a pressure of about 10,000 kilogrammes per square millimetre, we see that there is still a comparatively wide range of play for the magnitude of the tension.

Let us now establish the boundary conditions for the mantle of the cylindrical string. Since the string, apart from the two ends, which we suppose to be tightly clamped, is to vibrate freely, no external forces act on the mantle surface and hence we have for it, by (74) :

$$X_x \cos(\nu x) + X_y \cos(\nu y) + X_z \cos(\nu z) = 0, \ldots \quad . \quad (149)$$

The normal ν of the mantle surface may point in very different directions, but it is bound by the condition that it must everywhere form a right-angle with that curve in space which the string forms at any time. This space curve is formed by the points with the co-ordinates $x + u$, v, w and hence their direction ratios are :

$$d(x+u) : dv : dw = \left(1 + \frac{\partial u}{\partial x}\right) : \frac{\partial v}{\partial x} : \frac{\partial w}{\partial x}$$

and therefore the following condition holds :

$$\left(1 + \frac{\partial u}{\partial x}\right) \cos (\nu x) + \frac{\partial v}{\partial x} \cos (\nu y) + \frac{\partial w}{\partial x} \cos (\nu z) = 0$$

Since the deformations are infinitely small, $\cos (\nu x)$ is accordingly infinitely small compared with the other two cosines, as is easy to understand, and we may write more simply :

$$\cos (\nu x) = - \frac{\partial v}{\partial x} \cos (\nu y) - \frac{\partial w}{\partial x} \cos (\nu z)$$

If we substitute this value in the three equations (149) and reflect that the ratio $\cos (\nu y) : \cos (\nu z)$ can have any arbitrary value, we obtain, if we neglect small terms of the second order, the relationships :

$$\left.\begin{aligned} X_y &= X_x \frac{\partial v}{\partial x} = - \frac{F}{q} \frac{\partial v}{\partial x} \\ X_z &= X_x \frac{\partial w}{\partial x} = - \frac{F}{q} \frac{\partial w}{\partial x} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (150)$$

$$Y_y = 0, \; Z_z = 0, \; Z_y = 0 \quad . \quad . \quad . \quad (151)$$

It only remains to express also the sixth pressure component X_x accurately as far as terms of the second order. This is done by means of equations (148), which in combination with (151) give :

$$y_y = z_z = - \frac{\lambda}{2\mu} \sigma$$

from which, in view of (118), we get :

$$\sigma = \frac{\mu}{\lambda + \mu} x_x$$

and finally, by the first of the equations (148) together with (135) :

$$X_x = - \frac{F}{q} - E \frac{\partial u}{\partial x} \quad . \quad . \quad . \quad . \quad (152)$$

So we obtain the required equations of motion of the

string from (147) with the values (150), (151), (152) of the pressure components in the following form:

$$k\frac{\partial^2 u}{\partial t^2} - E\frac{\partial^2 u}{\partial x^2} = 0 \quad . \quad . \quad . \quad . \quad (153a)$$

$$k\frac{\partial^2 v}{\partial t^2} - \frac{F}{q}\frac{\partial^2 v}{\partial x^2} = 0 \quad . \quad . \quad . \quad . \quad (153b)$$

$$k\frac{\partial^2 w}{\partial t^2} - \frac{F}{q}\frac{\partial^2 w}{\partial x^2} = 0 \quad . \quad . \quad . \quad . \quad (153c)$$

Thus each of the displacement components u, v, w obeys its own particular law, independently of the other two, and the form of these laws is the same for all three components. But the value of the characteristic constant that occurs in the equations is not everywhere the same; a different value holds for u than holds for v and w. This of course arises from the fact that the direction of u coincides with the direction of the string, whereas the directions of v and w are perpendicular to it. Hence we call the u-vibrations *longitudinal vibrations* and the v- and w-vibrations *transverse vibrations* of the string. The longitudinal vibrations are caused, as we shall see, simply by the elasticity of the substance of the string, and, in particular, by the modulus of linear elasticity of the substance, and they are independent of the tension, whereas in the case of transverse vibrations the opposite is true.

For the further treatment of the laws of vibration it is obviously sufficient to consider a single component. Since in acoustics the transverse vibrations are by far the more important, we shall link up the following considerations with equation (153b), which represents the plane transverse vibrations in the xy-plane. Introducing the constant:

$$a^2 = \frac{F}{kq} \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

we may write it in the form:

$$\frac{\partial^2 v}{\partial t^2} = a^2\frac{\partial^2 v}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

and when integrated it gives v as a function of the two independent variables x and t. If we keep x constant and allow t to vary we get the motion of a definite point of the string. If, however, we keep t constant and allow x to vary we get the form of the curve which the string forms at a definite moment of time. Accordingly $\frac{\partial^2 v}{\partial t^2}$ denotes the acceleration of a point of the string, $\frac{\partial^2 v}{\partial x^2}$ the curvature of the curve formed by the string, and the equation (155) states that the acceleration of any point of the string, and hence also the force that acts on it, are proportional to the curvature of the curve formed by the string at this point, as is easy to see.

§ 35. **Integration of the Equation of Motion.** To find the general integral of the partial differential equation (155) we introduce a new independent variable in place of the independent variables x and t as follows:

$$\xi = x + at, \eta = x - at \quad . \quad . \quad . \quad (156)$$

The following relationships hold for the transformation of the differential coefficients:

$$\left(\frac{\partial v}{\partial t}\right)_x = \left(\frac{\partial v}{\partial \xi}\right)_\eta \cdot \left(\frac{d\xi}{dt}\right)_x + \left(\frac{\partial v}{\partial \eta}\right)_\xi \cdot \left(\frac{\partial \eta}{\partial t}\right)_x = \frac{\partial v}{\partial \xi} \cdot a - \frac{\partial v}{\partial \eta} \cdot a$$

from which by repeating the operation:

$$\left(\frac{\partial^2 v}{\partial t^2}\right)_x = a^2 \frac{\partial^2 v}{\partial \xi^2} - 2a^2 \frac{\partial^2 v}{\partial \xi \partial \eta} + a^2 \frac{\partial^2 v}{\partial \eta^2}$$

and, analogously:

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_t = \frac{\partial^2 v}{\partial \xi^2} + 2 \frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}$$

If in equation (155) we now substitute the differential coefficients in the independent variables x and t by those in the independent variables ξ and η, it follows that:

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = 0$$

This equation states that $\frac{\partial v}{\partial \xi}$ depends only on ξ and that $\frac{\partial v}{\partial \eta}$ depends only on η.

Hence:
$$v = f(\xi) + g(\eta)$$
where f and g are any two functions of a single variable. Or, in view of (156):
$$v = f(x + at) + g(x - at) \quad . \quad . \quad . \quad (157)$$

This is the general integral of the differential equation (155). It is, of course, seen at once by direct substitution that this differential equation is satisfied by the expression (157), no matter how f and g may be chosen. We must, however, note the following point.

The restriction which is imposed by (157) on the value of v and which so enables it to satisfy the differential equation (155) is due to the fact that the independent variables x and t in the function f occur only in the combination $x + at$, and in the function g only in the combination $x - at$. These combined expressions are also called the "arguments" of the functions f and g. Each of these two functions depends only on its particular argument. Hence if we differentiate the function, no matter whether with respect to x or to t, we can first differentiate with respect to the argument and then differentiate the argument with respect to the variables in question. In this way we obtain from (157):
$$\frac{\partial v}{\partial x} = f' + g', \ \frac{\partial v}{\partial t} = f' \, . \, a - g' \, . \, a. \quad . \quad . \quad . \quad (157a)$$
where we have denoted the differential coefficients of f and g with respect to their arguments by f' and g'. By continuing the process we get:
$$\frac{\partial^2 v}{\partial x^2} = f'' + g'', \ \frac{\partial^2 v}{\partial t^2} = f'' \, . \, a^2 + g'' \, . \, a^2$$
and these values are found to satisfy the differential equation (155) perfectly generally.

The special form of the function f or g can never be discovered from the differential equation, but emerges only from the initial conditions and the boundary conditions of the vibrating string. Before we pass on, however, to take into account these conditions we shall investigate generally the special physical character which the expression (157), found for the displacement v, impresses on the motion of the string.

If we first take the special case where one of the two functions, say g, vanishes, so that:

$$v = f(x + at) \quad . \quad . \quad . \quad . \quad . \qquad (158)$$

then we have a motion of the string for which the circumstance is characteristic that the displacement v does not change, whereas x and t change, but in such a way that $x + at$ remains constant, that is:

$$dx + adt = 0 \text{ or } \frac{dx}{dt} = a$$

But the last equation denotes a motion of velocity a in the negative direction of the x-axis. Hence it follows that if we pass down the string in the negative direction of the x-axis with the velocity a, either with the eye or with a pointer, the point of the string which is followed will always have a perfectly definite displacement v. We may also express this as follows: every displacement v propagates itself unchanged with the velocity a in the negative direction of the x-axis. Hence the shape of the curve which the string forms at any moment remains always the same; it displaces itself continuously only as a whole in the manner indicated. Such a motion is called a *wave-motion*, the velocity of displacement is called the *velocity of propagation* of the wave. The velocity of propagation is to be distinguished radically from the corpuscular velocity of the points of the string with which it is in no wise connected inherently (cf. I, § 1). The form of the wave is determined by the function f; it can be quite arbitrary and need not, in particular,

be periodic. A simple and very striking picture of the motion is obtained if we draw the function f as a curve on a strip of paper, with the argument ξ as abscissa and the value f as ordinate, and then draw this paper strip with the velocity $-a$ along the string. At every moment the drawing then directly gives the picture of the string. For $t = 0$ the argument $\xi = x + at$ coincides with the abscissa x of a point of the string.

In an exactly corresponding way the particular solution:

$$v = g(x - at) \qquad . \quad . \quad . \quad . \qquad (159)$$

denotes a wave-motion with the same velocity of propagation a but moving in the positive direction of the x-axis. The general case (157) of the vibration accordingly consists of two waves which are in general different and which move in opposite directions with the same velocity of propagation a. If we draw each of the two waves as a curve on a strip of paper and move this paper along the string with the velocities $\pm a$, then at every moment the algebraic sum of any two opposite ordinates gives the momentary displacement v of the corresponding point on the string.

We shall next investigate how the forms of the two waves f and g are determined by the initial and boundary conditions of the vibration process and for this purpose we shall first treat the ideal case of an infinitely long string.

§ 36. **String Unlimited in both Directions.** If the string stretches from $x = -\infty$ to $x = +\infty$ it is only necessary to take into account the initial state in order to determine the motion completely, and we do not require to consider the boundary conditions.

Let the displacements and the velocities of all the points on the string at the time $t = 0$ be given, that is:

$$v_0 = F(x) \quad \text{and} \quad \left(\frac{\partial v}{\partial t}\right)_0 = \Phi(x) \qquad . \quad . \qquad (160)$$

where F and Φ denote two functions which are known

for all positive and negative values of x. Substituted in (157) and (157a) this gives:

$$f(x) + g(x) = F(x)$$
$$f'(x) - g'(x) = \frac{1}{a}\Phi(x)$$

and by integrating the last equation:

$$f(x) - g(x) = \frac{1}{a}\int_c^x \Phi(x)dx$$

Consequently:

$$\left.\begin{aligned} f(x) &= \frac{1}{2}F(x) + \frac{1}{2a}\int_c^x \Phi(x)dx \\ g(x) &= \frac{1}{2}F(x) - \frac{1}{2a}\int_c^x \Phi(x)dx \end{aligned}\right\} \quad . \quad . \quad (161)$$

The forms of the two waves and also the whole motion are completely given by these expressions of f and g. In f we have only to replace x by the argument $x + at$, and in g by the argument $x - at$. The indefiniteness contained in the arbitrary choice of the integration constant c is only apparent because in the value (157) of v, f and g occur only as a sum, and a change in c alters the value of f in a reverse direction to that of g. Hence we may set $c = 0$ without loss of generality.

Let us consider several special cases as examples; first the one where *all initial velocities are equal to zero*, as in the case of a plucked string—that is, one which is taken out of its position of equilibrium and then released. In this case $\Phi(x) = 0$, and so by (161):

$$f(x) = g(x) = \tfrac{1}{2}F(x) \quad . \quad . \quad . \quad . \quad (162)$$

Then the two waves are equal to each other, and each of the two wave-functions is equal to half of the amount of the initial displacement. From this we directly obtain the whole course of the motion. In Fig. 5 the uppermost line represents the picture of the curve of the string in the initial state. The disturbance of the equilibrium is here restricted to a limited portion of the string; sup-

pose it to have been produced by plucking the string at a point B with a little rod and at the same time holding it fast at two other points A and C on opposite sides of B. If the string is released without being given an initial velocity it vibrates in the manner above described; the outlines, shown below the initial state, of the equal wave-functions f and g move along in the sense of

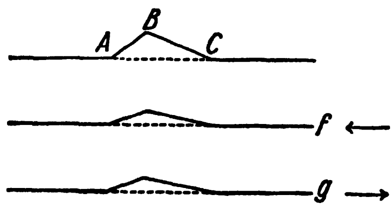

FIG. 5.

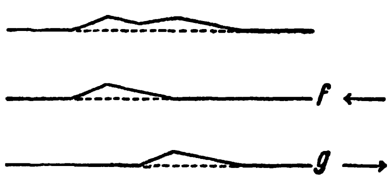

FIG. 5*a*.

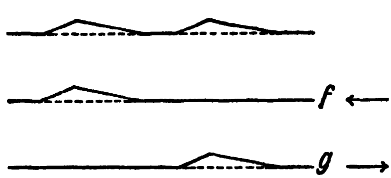

FIG. 5*b*.

the two arrows with the velocity a along the string, and the ordinates that lie above each other are algebraically added at each moment. Accordingly the original bulge in the string resolves into two bulges which are congruent, but only half as great, and which move apart with the velocities $\pm a$ in opposite directions, whereas between them repose again becomes established, as is depicted in Figs. 5a and 5b; in the top line of these figures the string is shown after a certain length of time has passed,

and in the two bottom lines the method of constructing the figures is given. There is no trace of periodicity in this process. So even the point B which has been furthest removed from the position of equilibrium returns directly to this position with constant velocity, and remains permanently at rest there.

The converse case where *all initial displacements are equal to zero* can be disposed of in an analogous way. An example of this is given by a long piano string which is struck so suddenly by a hammer at one point that the blow is ended before the struck point has perceptibly left its position of equilibrium. For then by (160):

$$F(x) = 0$$

and by (161):

$$f(x) = -g(x) = \frac{1}{2a}\int_c^x \Phi(x)dx \quad . \quad . \quad . \quad (163)$$

Thus the two wave-functions f and g are equal and opposite and by arguing further exactly as in the above case, we can show that two opposite and equal waves move apart in opposite directions from the point of impact, whereas the intervening part of the string at once returns to its state of rest.

But we must not believe that two waves move apart in both directions from the point of disturbance for *every* kind of disturbance of the equilibrium. To see this we shall propose the following question: what condition must be satisfied in the initial state in order that only one wave—say the f-wave—should be propagated from the point of disturbance? The answer is given by the equations (161) if we substitute $g = 0$ in it, thus:

$$F(x) = \frac{1}{a}\int_c^x \Phi(x)dx$$

or:

$$\frac{dF(x)}{dx} = \frac{1}{a}\Phi(x)$$

or by (160):

$$\frac{dv_0}{dx} = \frac{1}{a}\left(\frac{\partial v}{\partial t}\right)_0 \quad . \quad . \quad . \quad . \quad . \quad (164)$$

That is, in the initial state the velocity of a point on the string is equal to a times the tangent of the angle of inclination of the curve made by the string to the x-axis.

If this relationship between the velocity and the displacement is fulfilled for every point on the string, the whole disturbance moves in the direction of the negative x-axis as a single invariable wave; this can also be seen by a simple kinematic reflection, since the form of the curve f and the velocity of propagation a directly determine the velocity of every point on the string.

§ 37. **String Limited in both Directions.** We now consider the vibrations of a string of finite length l, which we shall suppose to be firmly clamped at the points $x = 0$ and $x = l$. Then of course the equations (161) again hold, but these equations have a meaning now only for those values of x which lie between 0 and l, because the functions $F(x)$ and $\Phi(x)$, which represent the initial values for the displacements and velocities of the points of the string, are defined only for $0 < x < l$. To represent the motion for *all* times, however, we require, by (157), the values of f and g also for other values of their arguments; for example, for great positive values of t we require the values of f for great positive values of the argument, and the values of g for great negative values of the argument. Hence we must supplement still further the determination of f and g given by the equations (161); this is made possible by the condition $v = 0$, which must be fulfilled at the boundaries $x = 0$ and $x = l$, or by (157):

$$0 = f(at) + g(-at)$$

and:

$$0 = f(l + at) + g(l - at)$$

must be valid for any arbitrary values of t. If we write x for at in these two equations, we get:

$$f(x) + g(-x) = 0 \quad . \quad . \quad . \quad . \quad (165)$$

$$f(l + x) + g(l - x) = 0 \quad . \quad . \quad . \quad (166)$$

These two equations which are valid for *all* values of x

constitute the necessary supplement, above mentioned, for calculating the wave-functions f and g. For if we next write $l + x$ in place of x in (166), it follows that:

$$f(2l + x) + g(-x) = 0$$

and, comparing this with (165), we get:

Likewise we get:

$$\left.\begin{aligned} f(2l + x) &= f(x) \\ g(2l + x) &= g(x) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (167)$$

That is, the wave-functions of f and g are both *periodic* with respect to x, having the period $2l$. By (157) it therefore follows directly that the motion of the string is also periodic with respect to the time t, the period being $\frac{2l}{a}$.

We shall first establish that the values of the wave-functions f and g are uniquely determined for all positive and negative values of their arguments by the initial conditions (161) and the boundary conditions (165) and (167).

In the first place, $f(x)$ and $g(x)$ are determined by (161) for $0<x<l$. Then the equations (165), used in the form:

$$f(x) = -g(-x) \text{ and } g(x) = -f(-x) . \quad . \quad (168)$$

give the values of $f(x)$ and $g(x)$ for $-l<x<0$. For the right-hand sides of these two equations are known for this range of values of x. But this means that $f(x)$ and $g(x)$ are given in the interval of a whole period—namely, from $x = -l$ to $x = +l$, and so (167) is completely determined.

We shall carry out this method of calculation for a special case. For simplicity we choose the vibration of a string in which initially only a single wave—say the f-wave—is present. We therefore suppose the relationship (164) to hold initially. Let the initial picture of the string be, say, that shown in Fig. 6. The wave-function $f(x)$ is then represented for $0<x<l$ by the same picture, whereas $g(x) = 0$ in this interval. On the other hand,

for $-l<x<0$ we have $f(x)=0$ throughout, whereas $g(x)$ has the form shown in the figure which we may call the "reversed image of $f(x)$ with respect to the point $x=0$." These forms repeat themselves periodically, as shown in Fig. 6.

These graphical representations of the wave-functions

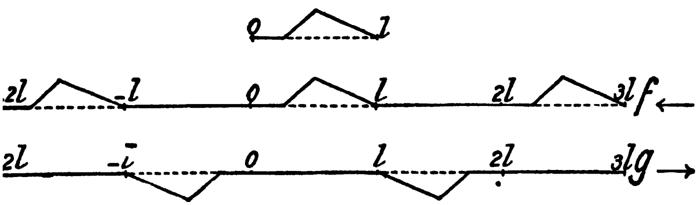

FIG. 6.

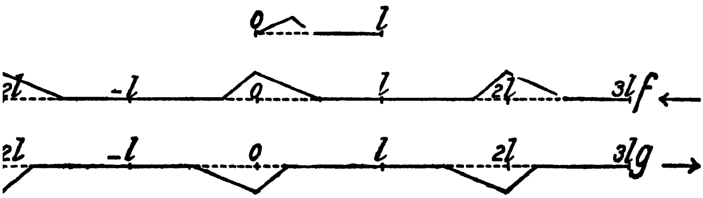

FIG. 6*a*.

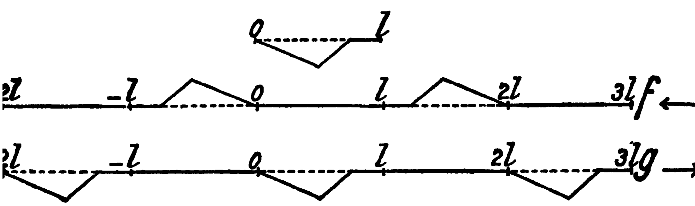

FIG. 6*b*.

f and g determine the whole course of the motion in the well-known manner. Figs. 6*a* and 6*b* again show in their uppermost lines the resultant graph of the string after a certain time has elapsed; in their lower lines they show how the graph is built up from f and g.

Accordingly, when the f-wave impinges on the fixed extremity $x=0$ of the string it becomes transformed into a g-wave which advances in the opposite direction—

that is, it is "reflected," its sign becoming reversed. After reflection the g-wave traverses the whole length of the string until it is reflected again at the other end, $x = l$, as an f-wave. After the period $\frac{2l}{a}$ has elapsed the old initial state is again reached and the process begins anew.

The general case, where the two waves, f and g, differ from zero, can be solved in the same way.

§ 38. **Analytical Representation of Periodic Functions.** In the above we have represented the periodic wave-functions $f(x)$ and $g(x)$ in the different ranges of x by means of various equations; but it is often also of value to denote a wave-function $f(x)$ in its whole course from $x = -\infty$ to $x = +\infty$ by means of a single analytical expression.

To accomplish this we first seek the most general analytical expression for a function $f(x)$ with the period $x = 2l$, or, in other words, the general solution of the functional equation (167). For this purpose we start out from the particular solution:

$$f(x) = e^{\alpha x} \quad . \quad . \quad . \quad . \quad . \quad (169)$$

which clearly satisfies the equation (167) if the constant α satisfies the condition:

$$e^{2l\alpha} = 1$$

whence it follows that:

$$2l\alpha = 2n\pi i$$

where n denotes any arbitrary positive or negative integer. If we substitute the value for α that results from this in (169) we obtain, by separating the real and the imaginary parts of $f(x)$, the two particular solutions:

$$\cos\frac{n\pi x}{l} \quad \text{and} \quad \sin\frac{n\pi x}{l}$$

which we can generalize still further by multiplying them by arbitrary constants A_n and B_n. If we form such solu-

tions for all possible order-numbers n where the A_n's and B_n's can have different values for each n, and add them together, we again obtain a solution of the functional equation (167) which, although we shall not prove it here, forms the *general* solution. It can be written in the following form:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}. \quad . \quad (170)$$

Here the terms involving negative values of n have been omitted and this, as we easily see, causes no loss of generality, since every term with a negative n may be combined with the term which contains the corresponding positive n to form a single term. Hence every term of the two sums essentially represents two terms. The term with $n = 0$ occupies a special position, as it occurs only once and so, as usual, has received the numerical factor $\frac{1}{2}$. This factor is also justified by the simpler significance which the constant A_0 then acquires, as will be shown below.

We recognize that the series (70), which is called "Fourier series" after its discoverer, actually has the period $2l$ in x by substituting $x + 2l$ for x, since then all the angles increase by whole multiples of 2π. But since *every* function which is periodic in x with the period $2l$ may be represented by (170), the question arises: how are the coefficients A and B calculated if the series of values of the function within a period is given—say, from $x = 0$ to $x = 2l$?

To answer this question we imagine the values of $f(x)$ to be given quite arbitrarily between $0<x<l$ and first calculate the integral:

$$\int_0^{2l} f(x) \, . \cos\frac{n\pi x}{l} dx \quad . \quad . \quad . \quad . \quad (171)$$

for any positive integer n that differs from zero. According to our hypothesis the integral has a perfectly definite value.

On the other hand, the integral (171), if $f(x)$ is replaced in it by the series (170), represents a sum of integrals which we now proceed to calculate. The first integral, which is multiplied by A_0, vanishes because:

$$\frac{A_0}{2}\cdot\int_0^{2l}\cos\frac{n\pi x}{l}dx = 0 \quad . \quad . \quad . \quad (172)$$

In the following integrals the order numbers 1, 2, 3 . . . n, . . . occur successively; in general they differ from the number n in (171), so we shall denote them by n'. If n' differs from n the corresponding two partial integrals are in each case:

$$A_{n'}\int_0^{2l}\cos\frac{n'\pi x}{l}\cdot\cos\frac{n\pi x}{l}dx$$
$$+ B_{n'}\int_0^{2l}\sin\frac{n'\pi x}{l}\cdot\cos\frac{n\pi x}{l}dx = 0 \quad . \quad . \quad (173)$$

But for the two terms in which $n' = n$ the corresponding integrals are:

$$A_n\int_0^{2l}\cos^2\frac{n\pi x}{l}dx + B_n\int_0^{2l}\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi x}{l}dx = A_n\,.\,l \quad (174)$$

Hence the total sum of the integrals into which (171) resolves reduces to the single expression (174), and we have the relationship:

$$\left.\begin{aligned} A_n &= \frac{1}{l}\int_0^{2l} f(x)\,.\,\cos\frac{n\pi x}{l}\cdot dx \\ &\text{In the same way:} \\ B_n &= \frac{1}{l}\int_0^{2l} f(x)\,.\,\sin\frac{n\pi x}{l}\cdot dx \end{aligned}\right\} \quad . \quad . \quad (175)$$

This determines the coefficients A_n and B_n completely. Finally, to obtain A_0, we get from (170):

$$\int_0^{2l} f(x)\,dx = \frac{A_0}{2}\cdot 2l = A_0\,.\,l$$

and see from this that the equations (175) may also be applied to the case $n = 0$. This shows us the advantage

of designating the constant by $\frac{A_0}{2}$. The value of B_0 is a matter of indifference, since $\sin\frac{2\pi n}{l}$ vanishes for $n = 0$.

From the method by which the constants A_n and B_n are calculated we also see that these constants are *uniquely* determined—that is, there is only one Fourier series of period $2l$, which assumes definitely prescribed values within the period. For if there were a second series—say with the coefficients A'_n and B'_n—these coefficients would also necessarily have to satisfy the equations (175); and since the expressions on the right-hand side of the

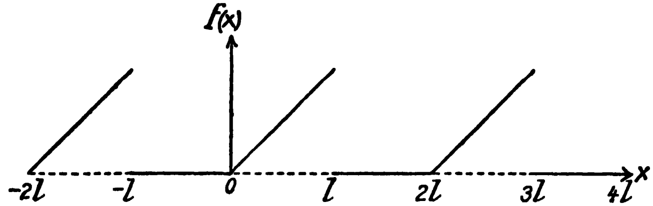

FIG. 7.

equations have prescribed values, so the coefficients become identical with the coefficients A_n and B_n.

To illustrate the power of Fourier expansions still more strikingly we shall perform the calculation for a special case. We shall assume that $f(x) = x$ for $0<x<l$, whereas for $l<x<2l$ $f(x) = 0$. The course of this function is represented by the curve in Fig. 7. In the ranges $-l$ to 0, from l to $2l$, and so forth it coincides with the axis of abscissæ, whereas in the intervening intervals it forms a part of a straight line which intersects the angle between the co-ordinates. Hence the function $f(x)$ is discontinuous for the x-values $-l, l, 3l$ . . . in that it jumps from l to 0 as x increases. The Fourier series which results from (175) and which we shall now set up has the same properties. The two integrals that occur in (175) may be resolved into two partial integrals, the first from 0 to l,

the second from l to $2l$. In the first integral $f(x) = x$, in the second $f(x) = 0$; accordingly:

$$A_n = \frac{1}{l}\int_0^l x \, . \cos\frac{n\pi x}{l}\,dx + 0 = \frac{(-1)^{n-1}}{n^2\pi^2} \, . \, l$$

$$B_n = \frac{1}{l}\int_0^l x \, . \sin\frac{n\pi x}{l}\,dx + 0 = \frac{(-1)^{n+1}}{n\pi} \, . \, l$$

whereas for $n = 0$ we get:

$$A_0 = \frac{1}{l}\int_0^l x\,dx = \frac{l}{2}$$

With these values of the coefficients the series (170) runs:

$$f(x) = \frac{l}{4} - \frac{2l}{\pi^2}\cos\frac{\pi x}{l} - \frac{2l}{9\pi^2}\cos\frac{3\pi x}{l} - \frac{2l}{25\pi^2}\cos\frac{5\pi x}{l} - \cdots$$
$$+ \frac{l}{\pi}\sin\frac{\pi x}{l} - \frac{l}{2\pi}\sin\frac{2\pi x}{l} + \frac{l}{3\pi}\sin\frac{3\pi x}{l} - \cdots, \qquad (176)$$

which is represented by the curve in Fig. 7 as a single analytical expression.

Let us take some special cases as tests. For $x = 0$ we have $f(x) = 0$, and hence:

$$0 = \frac{l}{4} - \frac{2l}{\pi^2} - \frac{2l}{9\pi^2} - \frac{2l}{25\pi^2} - \cdots,$$

or:

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots \qquad (177)$$

For $x = \frac{l}{2}$ we have $f(x) = x = \frac{l}{2}$, and so by (176):

$$\frac{l}{2} = \frac{l}{4} + \frac{l}{\pi} - \frac{l}{3\pi} + \frac{l}{5\pi} - \frac{l}{7\pi} + \cdots$$

or:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \qquad (178)$$

For $x = -\frac{l}{2}$ we have $f(x) = 0$, and so by (176):

$$0 = \frac{l}{4} - \frac{l}{\pi} + \frac{l}{3\pi} - \frac{l}{5\pi} + \frac{l}{2\pi} - \cdots$$

which again leads to the last series.

The behaviour of the Fourier series (176) is interesting at a point of discontinuity, such as $x = l$. Whereas the value of the series $= x$ or $= 0$ according as x approaches the value $x = l$ from smaller or greater values, for the value $x = l$ itself $f(x)$ is neither equal to l nor equal to 0, but, as substitution in (176) shows:

$$f(l) = \frac{l}{4} + \frac{2l}{\pi^2} + \frac{2l}{9\pi^2} + \frac{2l}{25\pi^2} + \cdots$$

or, by (177):

$$f(l) = \frac{l}{2}$$

—that is, the arithmetic mean of these two values. This theorem may be generalized, but we shall not enter into this question here.

§ 39. If, in particular, $f(x)$ is an "even" function—that is, if $f(-x) = f(x)$—it follows from (155), since the integration from 0 to $2l$ can be immediately replaced by that from $-l$ to $+l$, that:

$$A_n = \frac{1}{l}\int_l^0 f(x) \,.\, \cos\frac{n\pi x}{l}\cdot dx + \frac{1}{l}\int_0^l f(x) \,.\, \cos\frac{n\pi x}{l}\cdot dx$$

and for the first of these two partial integrals we get if we introduce the integration variable $x' = -x$:

$$-\frac{1}{l}\int_l^0 f(-x') \,.\, \cos\frac{n\pi x'}{l}\cdot dx' = \frac{1}{l}\int_0^l f(x') \,.\, \cos\frac{n\pi x'}{l}\cdot dx'$$

So the two partial integrals are equal and:

$$A_n = \frac{2}{l}\int_0^l f(x) \,.\, \cos\frac{n\pi x}{l}\cdot dx \quad . \quad . \quad . \quad (179)$$

In a corresponding way we get $B_n = 0$, since then the two partial integrals become equal and opposite.

Thus the expression of an even function in a Fourier series of period $2l$ runs more simply:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} \quad . \quad . \quad . \quad (180)$$

where the coefficients A are, by (179) uniquely determined by the course of the function within the half-period from

0 to l. Actually, every term of the series, as well as the constant A_0, is an even function of x.

If, on the other hand, $f(x)$ is an "odd" function—that is, if $f(-x) = -f(x)$—we get in the same way that $A_n = 0$ and:

$$B_n = \frac{2}{l}\int_0^l f(x) \, . \sin\frac{n\pi x}{l} \cdot dx \quad . \quad . \quad . \quad (181)$$

So that the Fourier series now runs:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \quad . \quad . \quad . \quad . \quad (182)$$

In this case, too, the coefficients are successively determined by the course of the function within the half-period.

§ 40. We now again revert to the problem of the vibrations of a string of length l and use the method developed in the last section to compress its solution—namely, the expression of the displacement v as a function of x and t into a single form which is valid for all values of x and t. By (157) we have:

$$v = f(x + at) + g(x - at)$$

Here the wave-function g can be generally reduced to f by means of the boundary condition (165) if we write $at - x$ in place of x in it:

$$f(at - x) + g(x - at) = 0$$

Thus:

$$v = f(x + at) - f(at - x)$$

Now by (167) $f(x)$ is periodic, having the period $2l$. Hence the equations hold for $f(x + at)$ and $f(at - x)$ which result if we write $at + x$ in (170) and then again $(at - x)$ in (170):

$$v = \frac{A_0}{2} + \Sigma A_n \cos\frac{n\pi}{l}(at + x) + \Sigma B_n \sin\frac{n\pi}{l}(at + x)$$

$$- \frac{A_0}{2} - \Sigma A_n \cos\frac{n\pi}{l}(at - x) - \Sigma B_n \sin\frac{n\pi}{l}(at - x) \, . \quad (183$$

The summations in each case are to be taken from $n = 1$ to $n = \infty$.

This is the general expression for the transverse vibrations of a string of length l. We can give it various simpler forms, each of which offers its special advantages. If we set:

$$A_n = C_n \cos \theta_n, \quad B_n = -C_n \sin \theta_n \quad . \quad . \quad (184)$$

with the further hypothesis that C_n is positive and that θ_n lies between 0 and 2π, we may write:

$$v = \sum_{n=1}^{\infty} C_n \cos \left\{ \frac{n\pi}{l}(at + x) + \theta_n \right\} - \sum_{n=1}^{\infty} C_n \cos \left\{ \frac{n\pi}{l}(at - x) + \theta_n \right\} \quad . \quad (185)$$

Thus the most general vibration of a string consists of a number of singly periodic waves, opposite and equal in pairs, and running to and fro, called "partial waves," each of which has the following period with respect to x:

$$\frac{2l}{n} = \lambda_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

and the period:

$$\frac{2l}{na} = \tau_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

with respect to t. The space-period λ_n is called the "wave-length," the time-period τ_n the "period or time of vibration" of the corresponding wave. The greatest possible wave-length (for $n = 1$) is $2l$—that is, twice the length of the string; corresponding to it is the longest time of vibration, $\frac{2l}{a}$. This vibration is called the "fundamental vibration" of the string. The other wave-lengths and times of vibration are the nth parts of the quantities that refer to the fundamental vibration.

The reciprocal value of the time of vibration:

$$\frac{1}{\tau_n} = \nu_n = \frac{na}{2l} \quad . \quad . \quad . \quad . \quad . \quad (188)$$

denotes the number of vibrations per unit of time and is therefore called the "frequency or vibration number" of the wave in question. According to the last three equations the wave-length, time of vibration and frequency are related as follows:

$$\frac{\lambda_n}{\tau_n} = \lambda_n \nu_n = a \quad . \quad . \quad . \quad . \quad . \quad (189)$$

If we introduce the wave-lengths and times of vibration equation (185) becomes:

$$v = \sum_{n=1}^{\infty} C_n \cos\left\{\left(\frac{t}{\tau_n} + \frac{x}{\lambda_n}\right)2\pi + \theta_n\right\}$$
$$- \sum_{n=1}^{\infty} C_n \cos\left\{\left(\frac{t}{\tau_n} - \frac{x}{\lambda_n}\right)2\pi + \theta_n\right\} \quad . \quad (190)$$

The whole angle contained in the large bracket is called the "phase" of the wave (I, § 12), the constant θ_n the "phase constant."

The expression for v becomes considerably simplified if in (183) or (190) we compress the terms corresponding to each order number n into a single term. It then follows from (183) that:

$$v = -2\sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi at}{l} - B_n \cos\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l} \quad (191)$$

This form of the equation of vibration is most appropriate for determining the values of the coefficients A_n and B_n from the initial state of the string. For at $t = 0$ we have by (160) and (191):

$$\left.\begin{aligned} v_0 &= F(x) = 2\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \\ \left(\frac{\partial v}{\partial t}\right)_0 &= \Phi(x) = -\frac{2\pi a}{l}\sum_{n=1}^{\infty} nA_n \sin\frac{n\pi x}{l} \end{aligned}\right\} . \quad (192)$$

The problem of determining A_n and B_n amounts to expanding the two given functions $F(x)$ and $\Phi(x)$ in Fourier sine series. But, as the equations (181) and (182) show, this is possible in only one way, since the

functions to be expanded are given just within a half-period of the series, from $x = 0$ to $x = l$. In (182) we have only to replace $f(x)$ by $F(x)$ or $\Phi(x)$, respectively, and B_n by $2B_n$ or $-\frac{2\pi a}{l}\,nA_n$, respectively. We then get from (181):

$$\left.\begin{aligned} 2B_n &= \frac{2}{l}\int_0^l F(x)\,.\,\sin\frac{n\pi x}{l}\,.\,dx \\ -\frac{2\pi a}{l}\,nA_n &= \frac{2}{l}\int_0^l \Phi(x)\,.\,\sin\frac{n\pi x}{l}\,.\,dx \end{aligned}\right\} \quad . \quad . \quad (193)$$

These equations determine the coefficients A_n and B_n for all values of n.

On the other hand, it follows from (185) by combining each two corresponding partial waves:

$$v = -2\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi at}{l} + \theta_n\right).\sin\frac{n\pi x}{l} \quad . \quad . \quad (194)$$

$$\frac{\partial v}{\partial t} = -\frac{2\pi a}{l}\sum_{n=1}^{\infty} nC_n \cos\left(\frac{n\pi at}{l} + \theta_n\right).\sin\frac{n\pi x}{l} \quad . \quad (195)$$

In this form the equation for the vibration of the string appears composed of a number of partial vibrations, each of which is represented as the sum of the two partial waves of the same order number, that move in opposite directions, and which therefore, in contrast with the "progressive waves," is called a "stationary or standing wave." If we consider a single member of these stationary waves, say of order number n:

$$\left.\begin{aligned} v &= -2C_n \sin\left(\frac{n\pi at}{l} + \theta_n\right)\sin\frac{n\pi x}{l} \\ &= -2C_n \sin\left(2\pi\nu_n t + \theta_n\right)\sin\frac{2\pi x}{\lambda_n} \end{aligned}\right\} \quad . \quad (196)$$

Taken alone it represents a possible vibration of the string; for, as we see, it fulfils the boundary condition $v = 0$ for $x = 0$ and $x = l$. Its peculiarity is that all points of the string are in the same phase. For the phase-angle which contains the time does not depend on

x, as it does in the case of the progressive wave. For this reason, all the points of the string pass through their position of equilibrium $v = 0$ simultaneously, and simultaneously attain their greatest elongation. The graph of the string is at every moment a sine curve, and the motion of every point a sine vibration, whose amplitude changes periodically from point to point. In the points :

$$x = 0, \frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}, \ldots \frac{n-1}{n} l, l \quad . \quad . \quad (196a)$$

the amplitude of vibration is zero. These points are therefore called the "nodes of the vibration." They divide the length of the string into n equal lengths of magnitude $\frac{l}{n}$, each of which is, by (186), equal to half the wave-length λ_n of the corresponding progressive

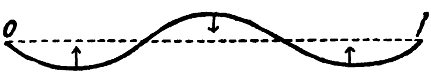

FIG. 8.

wave. In the case of the fundamental vibration ($n = 1$) the two extremities of the string are the only nodal points, in that of the second partial vibration ($n = 2$) a node lies in the middle of the string, and so forth. The vibrations occur between each pair of these nodes and in such a way that in adjacent sections on opposite sides of a node the vibrations are in opposite directions. The maxima of the amplitudes of vibration lie in the middle between the nodes and are called the "ventral segments or loops" of the vibration. The fundamental vibration has only a single ventral segment; in the same way, in the case of every other partial vibration the number of ventral segments is equal to the order number n of the vibration. Fig. 8 represents the graph of the string with the directions of the velocities for $n = 3$.

§ 41. The representation of the vibration of a string in the form (194) of singly periodic vibrations, however, besides being of interest on account of its mathematical

form, is also of great interest in physics and physiology on account of its importance for acoustics. For if the vibrations of the string are taken up by the air and transmitted through the ear to the ear-drum, the organ of hearing reacts to every single term (196) of this sum, and hence to every singly periodic vibration, by a particular sensation, that of the corresponding "partial tone." The vibration number ν_n determines the pitch of the tone, the amplitude constant C_n its intensity, whereas, on the other hand, the phase-constant θ_n has no acoustic significance whatsoever. According to the fundamental researches of Helmholtz, no specific physical attribute attaches to what is called in acoustics the quality or timbre of the tone. Rather, the timbre of a vibrating string is always to be traced back to the number ratios with which the amplitudes C_n of the individual partial tones occur in the vibration and which of course depend very essentially on the way in which the string is set into vibration. The more numerous and the more intense the partial tones are that occur, the sharper and the more shrill the tone becomes in general, whereas the fundamental tone ($n = 1$), or even an individual partial tone, when alone, sounds shallow and empty.

On account of these peculiar acoustic effects the partial vibrations and hence the individual members of the Fourier series of a wave also acquire an independent significance in physical respects, and we are therefore readily inclined to ascribe to them, both in the vibration of the string itself and also in the air-wave excited by the vibration of the string, an individual existence which is fully independent of the organ of hearing. But we must not forget that although such a conception is indeed admissible and to a great extent expedient it is by no means necessary. For so long as we fix our attention only on the vibration of the string and the air-wave, the dependence of the displacement v of a material point on the independent variables x and t, and hence all the peculiarities of the physical process, are completely

represented by a single function, and it is not possible without arbitrariness to resolve this function further into individual components. Thus the partial tones are not already contained in the vibration of the string or in the air-wave as real entities, but are separated out individually in the organ of audition.

The auditory nerve is, however, not only sensitive to the absolute value of a vibration number, but also reacts in a characteristic way to the ratio of the vibration numbers of two tone-waves which fall on it simultaneously by experiencing their joint effect as harmonious or consonant or pleasing when the ratio is simple and rational—that is, can be represented by the quotient of two small whole numbers: the absolute pitch of the tones are of no account in this connection. Hence in acoustics we measure the distance between tones, the musical "interval," not by the difference, but by the ratio of the vibration numbers or, what comes to the same thing, by the difference of their logarithms. The simpler this ratio the more perfect the consonance. Now since by (188) the vibration numbers ν_n of the partial tones of a vibrating string are in the ratio of the order-numbers $1:2:3:4:\ldots$, the tones of lower order number are consonant with the fundamental tone and with each other; hence they are called the "harmonic overtones."

The simplest interval is, if we except unison $(1:1)$ the ratio $2:1$—namely, the "octave." Here the joint effect of the two tones is so perfect that they often fuse into a single sensation and can be distinguished only with difficulty or not at all by practised ears. After the octave we have the ratio $3:2$ or the "fifth," and then $4:3$ or the "fourth." A fourth and a fifth together give an octave, for:

$$\frac{3}{2}\cdot\frac{4}{3}=2.$$

In general, an interval which converts another interval into an octave is called its "inversion." Hence the

fourth is the inversion of the fifth. Further, we have the ratio $5:4$ or the "major third" and its inversion $8:5$, the "minor sixth"; then the ratio $6:5$ or the "minor third" and its inversion $5:3$, the "major sixth." This completes the list of intervals usually called consonant, so far as we restrict ourselves to intervals that lie within an octave—that is, <2. Greater intervals are usually divided by an integral power of 2, so as to be less than 2.

To indicate the tones we usually divide their whole range into octaves, starting from an arbitrary normal tone which is chosen by common agreement. In accordance with the Paris tuning convention we start from the tone $\nu_0 = 261$, the once-accented c', which can be conveniently reached by men's and women's voices as well as by the most important instruments and thence we pass by octaves upwards to the c with five accents, c''''': $\nu = 261 \cdot 2^4$ and downwards to the small c, great C, contra C_1 and subcontra C_{11}: $\nu = 261 \cdot 2^{-4}$. Sound waves whose frequencies lie outside these limits produce no sensation of tone, but at the most act mechanically on the ear, producing a piercing effect when too high and a fluttering effect when too low. The range between the once-accented c, c', and the twice-accented c, c'' is called the once-accented octave, and in the same way every other octave bears the name of the lower c it contains. All tones which differ by an octave bear the same name, and the position of the octave is made recognizable in the same manner as that above used for the different c's.

Thus a string with the fundamental tone C (lowest tone on the violoncello) has the following overtones:

$n =$	1	2	3	4	5	6	7	8	9	10 ...
Partial tone	C	c	g	c'	e'	g'		c''	d''	e''

The overtones $n = 7, 11, 13, 14, \ldots$ are not used in modern music and hence have received no special names. Nevertheless we experience very clearly that the so-called

natural seventh 7 : 4 is occasionally superior to the diminished seventh 9 : 5, which is its nearest interval, when we have the opportunity of comparing these two intervals appropriately with each other. Without doubt the important part which the so-called dominant seventh chord plays in music (it consists of the fundamental tone, major third, perfect fifth and minor seventh) is due to the circumstance that the intervals of this chord are very nearly represented by the ratios 4 : 5 : 6 : 7.

§ 42. The consonant intervals form the foundation of all practical music. The quality of musical composition is bound to suffer if this natural law is disregarded. The totality of tones which are obtained if we start out from a normal tone ν_0 and pass through certain consonant intervals forms a *natural system of tones.* The frequencies of the tones that belong to a definite system may be represented by a single simple mathematical expression. If we restrict ourselves to octave intervals, the vibration numbers are of the form :

$$\nu = \nu_0 \,.\, 2^n \quad . \quad . \quad . \quad . \quad . \quad (197)$$

where n denotes any integral positive or negative number. These tones are distinguished only by octaves from the normal tone, and so they represent a manifold, which is insufficient for practical music. If, however, we also admit fifths, we obtain tones of frequencies :

$$\nu = \nu_0 \,.\, 2^n \,.\, 3^p \,. \quad . \quad . \quad . \quad . \quad . \quad (198)$$

where p likewise represents an integral positive or negative number. All tones, taken together, whose frequencies can be represented in the form (198) constitute the Pythagorean system of tones. This includes practically all tones that can be reproduced and hence also all intervals. For every arbitrary number may be represented to any degree of approximation, if not accurately, by (198). For example, the major third 5 : 4 can be obtained to an approximation which is to a certain extent useful by moving four fifth-intervals upwards and two octave-intervals

downwards—that is, by the interval $\left(\frac{3}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2 = \frac{81}{64}$, the so-called Pythagorean third, which is slightly greater than the natural third. The difference between these two thirds, $\frac{81}{64} \cdot \frac{4}{5} = \frac{81}{80}$, is called a "syntonic comma." By increasing the number of fifths and octaves we can of course get a closer approximation. In the absolute sense it is, however, impossible ever to reach the interval of a natural third by means of fifth or octave steps, no matter how many we take, and to this extent the Pythagorean system of tones is unsatisfactory. Theoretical requirements are better satisfied by means of the tone-system which has been enriched by means of steps of thirds:

$$\nu = \nu_0 \,.\, 2^n \,.\, 3^p \,.\, 5^q \quad . \quad . \quad . \quad . \quad (199)$$

and this has, in fact, long formed the basis of music.

Nevertheless even this system, as indeed all natural systems of tones, suffers from the serious disability that, strictly speaking, it actually comprises an unlimited number of tones, whereas practical music, in particular since the introduction of instruments with fixed tones (organs, pianos), has to do with a finite and not an infinite number of tones. But if we cut off the system of tones arbitrarily at definite limiting values of the numbers n, p, q—that is, after a definite number of intervals—the tones at the limits are at a disadvantage, since it is impossible to proceed further from them, whereas an unlimited power of modulation is one of the principal desiderata of the more recent music. This weakness was felt in the course of time to be impossible to bear, so that a radical alteration was decided on, which sacrificed the natural tone system in changing the consonant intervals slightly, or in "tempering" them. Among all the artificial tone systems the so-called *equally-tempered system*, consisting of twelve steps, has been found to be easily the most useful and has

established itself firmly since its introduction by J. S. Bach. In this system all octaves, since they are the most important consonances, are absolutely pure; the fifths are, however, tempered, use being made of the circumstance that twelve fifths in succession—that is, $\left(\frac{3}{2}\right)^{12}$—are nearly equal to seven octaves—that is, 2^7. The difference amounts to $\left(\frac{3}{2}\right)^{12} \cdot \left(\frac{1}{2}\right)^7 = \frac{74}{73}$, approximately, which is a "Pythagorean comma." If we now reduce all these twelve fifths uniformly and to such an extent that they together form exactly seven octaves, we obtain for the interval x of a tempered fifth the equation of condition:

$$x^{12} \cdot \left(\frac{1}{2}\right)^7 = 1 \quad . \quad . \quad . \quad . \quad . \qquad (200)$$

or:

$$x = 2^{\frac{7}{12}} = 1{\cdot}498$$

The difference between a pure and a tempered fifth accordingly amounts to only:

$$1{\cdot}5 : 1{\cdot}498 = \text{approximately } \frac{886}{885}$$

and is not directly perceptible by even the best-trained ear.

If we now arrange the twelve tones, at which we arrive by means of steps of twelve tempered fifths, into one octave—say the octave of small c's from c to c'—by transposing them down an appropriate number of octaves, then this octave is divided into twelve equal intervals according to the scheme:

2^0	$2^{\frac{1}{12}}$	$2^{\frac{2}{12}}$	$2^{\frac{3}{12}}$	$2^{\frac{4}{12}}$	$2^{\frac{5}{12}}$	$2^{\frac{6}{12}}$	$2^{\frac{7}{12}}$	$2^{\frac{8}{12}}$	$2^{\frac{9}{12}}$	$2^{\frac{10}{12}}$	$2^{\frac{11}{12}}$	$2^{\frac{12}{12}}$
c	c♯ = d♭	d	d♯ = e♭	e	f	f♯ = g♭	g	g♯ = a♭	a	a♯ = b♭	b	c′

and the tones defined in this way constitute the equally-tempered system, which now repeats itself in every octave in exactly the same way.

The interval between two neighbouring tones is the tempered semi-tone :

$$2^{\frac{1}{12}} = 1{\cdot}0595$$

The tempered major third consists of four semitones : it is greater than the natural third by the amount :

$$\sqrt[3]{2} \cdot \frac{4}{5} = \frac{126}{125} \text{ approx.}$$

but less than the Pythagorean third, since the interval just calculated is not as great as the amount of a syntonic comma $\frac{81}{80}$.

In the theory of consonant intervals a disinclination, which is easy to understand, against consciously giving up exact agreement with Nature in the matter of intervals, has sometimes taken the form of an attack on the equally-tempered tuning and has given rise to the demand for a return to natural tuning. We must bear in mind, however, that the question of the justification of any particular method of tuning in practical music is not one that is to be settled before the tribunal of science. In art it is not theoretical grounds but solely and alone the actual æsthetic effect that decides. Hence that system of tones will always occupy the first place which has at its disposal the most effective means of artistic expression. So natural tuning will acquire practical importance again only if and when a master-composer arises who will have more and deeper things to express than is possible in any other system of tuning.

§ 43. Having made this digression into the domain of art, we now again consider general acoustics, and in the first place direct our attention to the remarkable fact that the ear, in being able to hear the individual partial tones in a sound wave that impinges on it, essentially performs exactly the same as the mathematician when from a given function which is periodic but otherwise arbitrarily complicated he calculates the individual

components A_n and B_n of the corresponding Fourier series according to equations (175) by means of the integrations there prescribed. For the stimuli to which the auditory nerve responds are contained in every detail in the manner in which the ear-drum is excited by the impinging sound-wave. Thus they are exhaustively represented by a single function of the time $f(t)$, which gives the displacement of the ear-drum from its position of equilibrium at any moment, and in the ear the resolution of the sound (chord) into its partial tones is equivalent to representing $f(t)$ as a Fourier series. An idea of this power of the organ of hearing, which borders on the miraculous, may be gathered if we reflect that the trained ear of a conductor is able to distinguish in the mass of sound produced by a combined choir and orchestra not only the tones and qualities of the individual instruments, but also the individual letters of the words that are sung. In this respect the ear is infinitely superior to the idea. For a colour, white, green or blue, is always experienced as something uniform and we are unable to specify directly whether and how this colour is composed physically of other colours.

The following fundamental question now arises: how is the ear able to accomplish such feats and what physical processes underlie the resolution of a musical sound into its partial tones? Helmholtz has given an answer to this question by starting out from the assumption that the subjective process which consists in experiencing a definite partial tone of frequency ν is always connected with a definite objective process—namely the pendulum-like vibration of a certain elementary material configuration in the inner ear with the same frequency ν, and conversely. Corresponding to all the different partial tones that the ear may perceive there is an exactly equal number of such elementary pendulums in the interior of the ear, arrayed alongside one another like the strings of a harp. Now if the ear-drum is caused to vibrate owing to the action of an incident air-wave $f(t)$, the partial

tone ν is heard or not according to whether the elementary pendulum which is characterized by the frequency ν is made to vibrate appreciably or not by the trembling of the ear-drum. The question as to whether the partial tone ν is contained in a sound wave $f(t)$ therefore amounts essentially to the question as to whether a force of the intensity $f(t)$ is able to cause a pendulum of period $\tau = \frac{1}{\nu}$ to vibrate appreciably or not. But this question is accurately answered in general mechanics—namely, in the following way.

By I, § 13, a pendulum of a definite period of vibration $\tau = \frac{1}{\nu}$ is represented by a material point which executes small oscillations about a stable position of equilibrium $x = 0$. If the vibrations are caused by a single impact, but otherwise occur without external disturbance, they are represented by the equation of motion:

$$m\frac{d^2x}{dt^2} = -4\pi^2 m\nu^2 x \quad . \quad . \quad . \quad . \quad (201)$$

which results from I, (15) if we replace the constant c by the frequency ν, according to the relationship:

$$2\pi\sqrt{\frac{m}{c}} = \tau = \frac{1}{\nu} \quad . \quad . \quad . \quad . \quad (202)$$

In this case the pendulum executes "free" vibrations, whose period τ is therefore also called the "proper or natural period" of the pendulum.

But if, in addition, an external force $f(t)$ acts on the pendulum, it executes "forced" vibrations in accordance with the equation of motion:

$$m\frac{d^2x}{dt^2} = -4\pi^2 m\nu^2 x + f(t) \quad . \quad . \quad . \quad (203)$$

If we first calculate the forced vibrations for the special case where the external force $f(t)$ is singly periodic—that is, of the form:

$$f(t) = C\cos(2\pi\nu' t + \theta) \quad . \quad . \quad . \quad (204)$$

with the further assumption that in the initial state, $t = 0$, the pendulum is at rest in its position of equilibrium —that is, $x = 0$ and $\frac{dx}{dt} = 0$.

The general solution of the differential equation:

$$m\frac{d^2x}{dt^2} + 4\pi^2 m\nu^2 x = C\cos(2\pi\nu' t + \theta)\,. \quad . \quad (205)$$

is of the form:

$$x = \alpha\cos(2\pi\nu t + \theta'_0) + \beta\cos(2\pi\nu' t + \theta)\,. \quad (206)$$

For if we substitute this expression for x in (205) the left-hand side of the equation reduces to terms in β, and we get:

$$\beta = \frac{C}{4\pi^2 m(\nu^2 - \nu'^2)} \quad . \quad . \quad . \quad . \quad (207)$$

while α and θ_0 remain undetermined and therefore represent the two constants of integration. If we adapt the values to the initial state and then substitute them together with the value for β in (206), we get the desired pendulum vibration:

$$x = \frac{C}{4\pi^2 m(\nu^2 - \nu'^2)} \times$$

$$\left\{-\cos\theta\cos 2\pi\nu t + \frac{\nu'}{\nu}\sin\theta\sin 2\pi\nu t + \cos(2\pi\nu' t + \theta)\right\} \quad (208)$$

Thus the pendulum will be disturbed from its position of rest and will be made to perform a motion which will in general consist of vibrations of two different periods whose frequencies ν and ν' correspond to the proper or natural vibration and the external exciting vibration.

But there is an exception in the case where the period of the exciting vibration agrees with that of the natural vibration—that is, when $\nu' = \nu$. For then the expression (208) assumes the form $\frac{0}{0}$, and we arrive at its true value by setting $\nu' = \nu + \Delta\nu$ and then expanding the numerator

and the denominator in powers of $\Delta\nu$, finally passing to the limit $\Delta\nu = 0$:

$$x = \frac{C}{8\pi^2 m\nu^2} \cdot \{2\pi\nu t \sin(2\pi\nu t + \theta) - \sin 2\pi\nu t \sin\theta\} \quad . \quad (209)$$

We can, of course, easily convince ourselves subsequently that the expression (209) actually satisfies both the initial conditions and the differential equation (205) for $\nu' = \nu$.

On account of the factor t which occurs before the sine terms this motion of the pendulum is totally different from that represented by (208), not only in representing a different law, but also in its order of magnitude. For it does not consist in periodic vibrations, but increases to an infinite extent with time. Thence it follows that an external periodic force of no matter how small an amplitude C, provided only that its period agrees with the natural period of the pendulum, produces a greater effect in the course of time than any arbitrarily great force of any other period. In the case envisaged we say that the exciting force is in *resonance* with the pendulum. Resonance provides the explanation for numerous striking natural phenomena, of course, not alone in the domain of acoustics. If a weak air wave causes the prongs of a large tuning-fork to vibrate, or when a small boy causes a heavy church-bell weighing some hundredweights to vibrate intensely, or when a strong bridge begins to oscillate dangerously owing to the regular footsteps of those who traverse it, these are all effects of resonance, and it suggests itself to us to assume that, as in acoustics and electrodynamics, so also in chemistry and even in biology the violence of many reactions is to be traced back essentially to resonance.

If we now assume more generally that the force that excited the pendulum is not singly periodic, but is composed of a series of singly periodic vibrations—that is, is expressed by a Fourier series, which, as we know, at the same time represents the most general case—the vibrations of the pendulum will or will not in the course

of time increase to an infinite extent, according as the Fourier series contains a term involving the natural period of the pendulum or not. The value of the coefficient of this term in relation to the other terms of the Fourier series does not enter into the question at all. Thus the pendulum shows by the manner in which it responds to the exciting vibration whether the corresponding term in the Fourier series is present or not, and by its resonance it serves to analyse the series, and hence is also called a *resonator.* A resonator responds appreciably only to the vibration which corresponds with its natural period; hence its action is *selective.* The action of resonance also gives the individual terms of a Fourier series a physical meaning, whereas hitherto only their sum could be defined physically.

Let us now investigate more closely the quantitative relationship between the exciting force $f(t)$ and the energy of the vibration which is excited in the resonator. We consider the process during a certain interval of time and assume the exciting force $f(t)$ to be so weak that during this interval of time the pendulum vibrates almost freely—that is, periodically. Then for an interval of time dt the change in the energy of vibration E (sum of the kinetic and potential energies) is, by I (393):

$$dE = F_x \cdot dx = f(t) \cdot dx$$

or:

$$dE = f(t) \cdot \frac{dx}{dt} \cdot dt$$

Thus the change of energy in the time from t to t' is:

$$E' - E = \int_t^{t'} f(t) \cdot \frac{dx}{dt} \cdot dt \quad . \quad . \quad . \quad (210)$$

According as the displacements x of the pendulum during the interval of time under consideration are represented by $\cos 2\pi\nu t$ or by $\sin 2\pi\nu t$, $\frac{dx}{dt}$ is proportional to $\sin 2\pi\nu t$ or $\cos 2\pi\nu t$, and the integrals (210) assume exactly the same forms as the integrals (175), which

serve to calculate the coefficients of the Fourier series for the function $f(t)$, except that the integration variable was there denoted by x and here by t. Thus if we picture to ourselves a long series of such resonator pendulums of appropriate periods and phases of vibration placed alongside one another and all subjected to the same exciting force $f(t)$, each pendulum indicates by the energy of its vibration the magnitude of the corresponding coefficient in the Fourier expansion of $f(t)$, and, according to Helmholtz, this process is in essence identical with that which the ear performs when each of the auditory threads which responds to a definite frequency takes up the vibrations of the ear-drum by the action of resonance and passes it on to the nerves of sensation.

From this we not only see, as was explained at the beginning of this section, that the sense of hearing carries out the same resolution of the sound-wave as corresponds to the transformation of this wave into a sum of singly periodic waves, but that Nature makes use of the same method in this process as mathematics. For the last equation (210) is tantamount to a mechanical illustration of the mathematical definition of the series coefficients in (175). The recognition of this fact completely accounts physically for the ability of the ear to resolve a musical sound into its partial tones.

Another question of fundamental importance is that which is concerned with the peculiar effect of chords of consonant intervals. Helmholtz has also developed a theory for this, according to which the concept of consonance is not due to a certain unconscious pleasure of mystic origin, which the auditory nerve experiences when simple rational number ratios present themselves, but that here, too, perfectly definite real states that are physically definable play the decisive part, which in the case of consonance induce pleasure and in the case of dissonance displeasure. It will be better to leave a short discussion of these matters until the next chapter, where we shall discuss acoustic waves in air (§ 46).

CHAPTER IV

VIBRATIONS IN LIQUIDS AND GASES

§ 44. AMONG perfectly elastic isotropic bodies (§ 21) liquids and gases are distinguished as forming a special class, since for them the three principal pressures (§ 20) are equal to each other, whence it follows that a normal pressure $X_x = Y_y = Z_z = p$ acts on each surface-element, which is independent of the orientation of the surface-element, being positive when the body is subject to compression. Since the tangential pressures vanish, the elastic constant μ in equations (119) also vanishes, and the pressure depends on the volume alone, or, respectively, on the density k:

$$p = f(k) \quad . \quad . \quad . \quad . \quad . \quad (211)$$

The form of the function f depends on the nature of the body. In the case of liquids that break into drops p varies particularly strongly with k, so that for these it is better to express k as a function of p. The limiting case is given by incompressible liquids for which k is constant. We shall discuss the form of f in detail later when we deal with finite deformations (§ 56).

Since in the interests of generality the following investigations are extended to liquids that break into drops, as well as to gases, it is advantageous to make use of an extension of the equations (119), which we introduced in § 33, by assuming the pressure p not to be equal to zero in the undeformed state ($\sigma = 0$), but rather equal to p_0, since a gas which is in stable equilibrium always exerts a finite pressure. The six equations (119) then reduce to the single equation:

$$p = p_0 - \lambda\sigma \quad . \quad . \quad . \quad . \quad . \quad (212)$$

The coefficient of elasticity λ is dependent on the compressibility, since:

$$\frac{-\sigma}{p - p_0} = \frac{1}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (213)$$

and may be calculated for every state of the substance if the function f in (211) is given. For since the mass of a body-element of volume dV is invariable, we get for an infinitely small volume dilatation σ the relationship:

$$dV \, . \, k = dV\,(1 + \sigma) \, . \, (k + dk) \quad . \quad . \quad (213a)$$

that is:

$$\sigma = -\frac{dk}{k}$$

and from (213):

$$\lambda = k\frac{dp}{dk} \quad . \quad . \quad . \quad . \quad . \quad . \quad (214)$$

Since the liquid and the gaseous states may be regarded as special cases of the solid state, we shall borrow the equations of motion directly from those for solid bodies, again disregarding gravitational forces. We then get from (147) and (212):

$$k\frac{\partial^2 u}{\partial t^2} = \lambda\frac{\partial \sigma}{\partial x}, \; k\frac{\partial^2 v}{\partial t^2} = \lambda\frac{\partial \sigma}{\partial y}, \; k\frac{\partial^2 w}{\partial t^2} = \lambda\frac{\partial \sigma}{\partial z}$$

or in vector form, if we write:

$$\frac{\lambda}{k} = \frac{dp}{dk} = a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (215)$$

we get:

$$\ddot{\boldsymbol{q}} = a^2 \, . \, \mathrm{grad}\,\sigma \quad . \quad . \quad . \quad . \quad . \quad (216)$$

If we apply the vector operation curl (§ 13) to this equation, then in view of (65):

$$\ddot{\boldsymbol{o}} = 0$$

and, integrating this equation twice with respect to t:

$$\boldsymbol{o} = \boldsymbol{c}_1 t + \boldsymbol{c}_2$$

That is, the components ξ, η, ζ of the rotation of a mass-particle are linear functions of the time. Hence if the

vector c_1 is not equal to zero, the rotation will increase by an infinite amount with the time. Since we wish to deal here with infinitely small oscillations about the position of equilibrium, a constant rotation in time does not interest us and so we restrict ourselves to considering the case where both constants c_1 and c_2 vanish, so that:

$$o = \operatorname{curl} q = 0 \quad . \quad . \quad . \quad . \quad (217)$$

The general solution of this differential equation is:

$$q = -\operatorname{grad} \phi . \quad . \quad . \quad . \quad . \quad (218)$$

where ϕ denotes any scalar function of the position and the time. Since this function bears the same relationship to the displacement vector q as, by I, (121), the potential U to the force-vector F, we may also call ϕ the "displacement potential." The existence of a displacement potential is therefore tantamount to stating that no rotations occur in the displacement. We see from (218) that in the quantity ϕ of the displacement potential an additive time-function is totally devoid of physical meaning, and hence may be defined only by an arbitrary convention.

If the displacement potential ϕ is known as a function of x, y, z, t all the details of the motion follow from it uniquely. For from (218) we get the velocity:

$$\dot{q} = -\operatorname{grad} \dot{\phi} . \quad . \quad . \quad . \quad . \quad (219)$$

So the velocity, too, has a potential, the "velocity potential." Further, from (66) it follows that for the volume dilatation:

$$\sigma = \operatorname{div} q = -\operatorname{div} \operatorname{grad} \phi = -\Delta\phi . \quad . \quad (220)$$

and for the fluctuation of pressure, from (212):

$$p - p_0 = -\lambda\sigma = \lambda . \Delta\phi \quad . \quad . \quad . \quad (221)$$

Hence the whole problem reduces to finding the displacement potential ϕ. For this purpose we make use

of the following differential equation, which follows from the equations of motion (216):

$$\text{grad}\,\ddot{\phi} = a^2 \,.\, \text{grad}\,\Delta\phi$$

which, when integrated with respect to the three co-ordinate directions, gives:

$$\ddot{\phi} = a^2 \,.\, \Delta\phi \quad . \quad . \quad . \quad . \quad . \quad (222)$$

Any constant of integration that presents itself can depend only on the time, and hence may forthwith be omitted as being of no physical significance.

The equation (222) which, as we may easily convince ourselves by performing the differentiations, holds not only for the displacement potential ϕ, but also for every component of the displacement and of the velocity, as well as for the volume dilatation σ and for the pressure p, represents a generalization of equation (155) for the vibrations of a stretched string, and hence is also called the spatial "wave-equation." The constant a of course here again plays the part of the velocity of propagation. Nevertheless the waves which are now under consideration are fundamentally different from those which occur when a string vibrates transversely, inasmuch as here the elastic effects are produced, not by rotations, but by changes of density and pressure.

As has already been emphasized in dealing with the problems of statics, we are always less interested, from the physical point of view, in the general solution of the characteristic differential equation than in those particular solutions which are adapted to certain important processes in Nature. We shall now consider a few of these.

§ 45. The simplest particular solution of the differential equation (222) is given by the special case where the function ϕ depends only on a single space-variable x. For the equation (222) then reduces to the simple form (155) and hence represents two waves which advance with the velocity a in the directions of the positive and the negative x-axis. Since the physical state of all the

material particles which lie in a plane perpendicular to the x-axis is the same, these waves are called *plane waves* and every plane $x = \text{const.}$ is called a "wave-plane" and the direction x which is perpendicular to it is called the "wave-normal." The form of the waves is determined by the initial state and the boundary conditions. Since the displacement potential ϕ depends, apart from the time, only on x, of the three displacement components only u differs from zero; thus the vibrations occur in the direction of propagation and the waves are, in contrast with the transversal vibrations of strings, which were treated in the preceding chapter, longitudinal.

Plane waves may be realized approximately in practice, for example, by enclosing a liquid or a gas—say air—in a long cylindrical tube. The tube then serves as a "pipe," the normal cross-sections are the wave-planes, and the vibrations of the air take place only in the longitudinal direction, which we take as our x-axis, according to the same laws as we discovered above for the vibrations of strings. It is therefore superfluous to enter into the details again here. Only the boundary conditions that are introduced require special comment. The end of a pipe can be open or closed. In the latter case $u = 0$ at the closed end because the air can neither penetrate into the wall of the vessel nor move away from it. This boundary condition therefore corresponds exactly with that which prevails at the end of a string which is fastened at one end. But in the former case we have at the open cross-section of the open end, according to the principle of action and reaction (Newton's third law), that the pressure p of the air, which vibrates in the pipe, on the external atmospheric air is equal to the pressure of the atmosphere on the air in the pipe—that is, it is constant and equal to p_0; in other words, $\sigma = 0$, or the density is constant in time. This leads uniquely to the laws which govern the vibrations of the air in open and closed (roofed) pipes. If, first, the pipe is *closed at both ends*, then we have, as in § 40, stationary vibrations of the form

(194), where we have now to regard v as having been replaced by u. The vibrations can be excited either by an external shock or by the use of a little hole for blowing into. At each end of the pipe there is a node of the displacement u and of the velocity $\dot{u}$. The position of the remaining nodes for the nth partial vibration (harmonic) is represented in (196a). An anti-node (also called loop or ventral segment) exists midway between two adjacent nodes. If we now inquire into the volume dilatation we get from (194):

$$\sigma = \frac{\partial u}{\partial x} = -\frac{2\pi}{l}\sum_{n=1}^{\infty} nC_n \sin\left(\frac{n\pi at}{l} + \theta_n\right) . \cos\frac{n\pi x}{l} . \qquad (223)$$

From this we see that the volume dilatation, and with it the fluctuation of pressure, has an anti-node at both ends of the pipe, and that the nodes of the volume dilatation: $\sigma = 0$—that is, the points where the density and pressure of the air are permanently constant—lie exactly at the anti-nodes of the displacement and of the velocity, or midway between their nodes.

If we now imagine the vibrating pipe to be severed at such a node of the dilatation $\sigma = 0$ and allow the open ends to be in free communication with the atmosphere, the course of the vibrations is nowhere disturbed in either of the two portions of pipe, since the boundary condition set up above is fulfilled at the opening. This gives us the laws which control the vibrations in a pipe open at both ends or in one open at only one end. If the pipe is *open at both ends*, the volume dilatation at each end has a node, and the displacement an anti-node. The series of partial tones is exactly the same as in a pipe which is closed at both ends or as in a string which is held at one end, except that the nodes and the anti-nodes have now changed places. In the case of the fundamental vibration, σ thus has a single anti-node and u a single node in the middle of the pipe.

But the law regulating partial tones is essentially different in a *roofed pipe*, which is open at one end and

closed at the other. In this case the diplacement u has a node at one end and an anti-node at the other. Hence in such a pipe the fundamental vibration has neither nodes nor anti-nodes in its interior; rather, in the fundamental vibration the whole length of the pipe is equal to the distance between a node and an anti-node—that is, it is equal to half the distance between two neighbouring nodes in a longer pipe if it executes the same stationary vibration. A pipe closed at one end therefore has the same fundamental vibration as a pipe of double the length which is either open at both ends or closed at both ends; and in the case of overtones in a closed pipe the length of pipe always includes an odd multiple of the distance between successive nodes and anti-nodes; hence it follows that the frequencies of the partial tones are in the ratio of the odd integers $1:3:5:7:\ldots$

§ 46. Even in the free atmosphere plane waves are approximately realized if the sound-wave occupies a comparatively small space and if the reference-point (I, § 35) is sufficiently far from the source of the sound. Then the distance x of the reference-point from the source represents the wave-normal and we have for a single tone of frequency $\nu = \frac{1}{\tau}$ and of wave-length λ, just as in (190), the following particular solution of (222):

$$\phi = C \cos \left\{ 2\pi \left(\nu t - \frac{x}{\lambda} \right) + \theta \right\} \quad . \quad . \quad (224)$$

where the relationships (189) hold between ν, τ and λ. By adding up several such particular solutions we obtain a more general solution, which corresponds to the case where several sound-waves simultaneously strike the ear from the same direction. A particularly interesting case is that in which two tones of the same intensity and only slightly different frequencies ν and ν' act conjointly. We then have for the vibration at a point which we shall for simplicity take as our origin $x = 0$, that:

$$\phi = C \cos (2\pi\nu t + \theta) + C \cos (2\pi\nu' t + \theta')$$

or:

$$\phi = 2C\cos\left(2\pi\cdot\frac{\nu'-\nu}{2}t + \frac{\theta'-\theta}{2}\right)\cdot\cos\left(2\pi\cdot\frac{\nu'+\nu}{2}t + \frac{\theta'+\theta}{2}\right) \quad . \quad . \quad (225)$$

If we now suppose $\nu' - \nu$ to be small compared with ν, then the first cosine in (225) changes but slowly with the time, and many vibrations of the single tones occur before its value changes appreciably. Hence the vibration (225) acts on the ear exactly like the vibration of a single sound-wave of variable amplitude:

$$C_t = 2C\cos\left(2\pi\cdot\frac{\nu'-\nu}{2}t + \frac{\theta'-\theta}{2}\right) \quad . \quad . \quad (226)$$

and of frequency $\frac{\nu'+\nu}{2}$, the arithmetic mean of the frequencies of the individual tones. What we hear, then, is a definite tone whose intensity fluctuates to and fro slowly and periodically between zero and a maximum. These fluctuations of intensity are called the "beats" of the two combining individual tones, and their maxima are called the "pulses" of the beats. The frequency with which these beats occur is determined by the time which elapses between two successive zero values of C_t, and hence their number amounts to $\nu' - \nu$ per unit of time. The closer together the individual tones lie the slower and the more easily distinguishable the beats become; if the tones move apart the beats accumulate and overlap, blurring each other. Hence beats offer an excellent means for testing by ear not only the interval between two simple and nearly equal tones, but also the duration of two musical tones that are nearly in accord from complete consonance. For in the case of two musical tones (notes) which are nearly consonant beats also occur—not, indeed, between the fundamental tones, but between individual pairs of the partial tones contained in the tones. For example, in the case of an octave which is out of tune the first overtone of the deeper note forms

beats with the fundamental of the higher note; in the case of a fifth which is out of tune the second overtone of the deeper note forms beats with the first overtone of the higher note. Hence it is much easier, comparatively, to tune an instrument in the natural system of tones than in the system of equal temperament.

According to Helmholtz it is these beats that form the physical grounds for the antithesis between consonance and dissonance. For it is a fact, which applies to all sense-organs, that intermittent stimuli, for a certain range of frequency, produce an irritating, fatiguing and generally unpleasant effect. We need only quote as examples reading in a flickering light or looking at certain kinds of illuminated advertisements. In the same way the auditory nerve suffers in the case of rattling, clattering, tremulating noises, if the frequency with which the sounds succeed each other attains a certain value. If this value is appreciably exceeded, the ear is no longer capable of distinguishing the individual stimuli and the feeling of discomfort vanishes; if, on the other hand, the beats become very slow, the ear is able to follow the swelling up and dying down of the intensity of the tone more easily and experiences a feeling of relief, akin sometimes to a feeling of alleviation from pain, at the moment when the interval between two tones which gradually approach each other more and more closely pass into unison and the beats become more and more rare until they become lost in the infinity of time.

Just as in the case of the unison of two tones, so also, in accordance with the above reflections, the same results apply to the coincidence of overtones of two consonant notes, and the unpleasing effect which we experience on hearing a dissonant interval, as well as the pleasure we feel in hearing a consonant interval, may also be brought into relationship with the appearance or disappearance of disturbing beats. This view appears to deprive harmony, as founded on consonance, of its mystic character and to trace back the particular nature of the cor-

responding auditory sensation to physical and physiological processes.

§ 47. Another comparatively simple solution of the wave-equation (222) is obtained if we assume that the displacement potential ϕ depends not only on the time, but also on the distance $r = \sqrt{x^2 + y^2 + z^2}$ of the reference-point (x, y, z) from the origin of co-ordinates. We then have by I, (110):

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{\partial \phi}{\partial r} \cdot \frac{x}{r}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial r^2} \cdot \frac{x^2}{r^2} + \frac{\partial \phi}{\partial r} \cdot \frac{1}{r} - \frac{\partial \phi}{\partial r} \cdot \frac{x^2}{r^3}$$

and corresponding expressions result for $\frac{\partial^2 \phi}{\partial y^2}$ and $\frac{\partial^2 \phi}{\partial z^2}$. This transforms $\Delta \phi$ into:

$$\Delta \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial^2}{\partial r^2}(r\phi) \quad . \quad . \quad (227)$$

and the wave-equation (222) may be written in the form:

$$\frac{\partial^2 (r\phi)}{\partial t^2} = a^2 \frac{\partial^2 (r\phi)}{\partial r^2} \quad . \quad . \quad . \quad . \quad (228)$$

Thus it has the same form as (155) and hence, by (157), has the same general solution:

$$r\phi = f(r + at) + g(r - at) \quad . \quad . \quad . \quad (229)$$

These are two spherical waves, of which the first travels inwards and the second outwards with the velocity a. But in this case it is not, as in the case of plane waves, the displacement potential itself which propagates itself unchanged, but its product with r.

Let us first consider as an example a spherical wave which propagates itself outwards—that is, we set $f = 0$. Then we get:

$$\phi = \frac{1}{r} \cdot g(r - at) \quad . \quad . \quad . \quad . \quad (230)$$

In order that the wave-function g may be finite, ϕ must become infinitely great when $r = 0$; so the origin

is a singular point of this wave. Hence when we realize the wave the origin must be excluded from the value of the vibrating fluid (air); actually it is there that the source of sound is situated. Let us imagine, for example, as the source of sound a sphere which is entirely surrounded by a fluid; the sphere is to consist of some solid elastic substance whose volume can be alternately increased and decreased according to some law by means of some mechanism in its interior—a so-called "pulsating sphere." This sphere transfers its vibrations to the immediately adjacent fluid, and this boundary condition determines the form of the wave-function g, since we can, on the other hand, also obtain the value of the displacement q at the boundary from the displacement potential by (218) and (230). The displacement occurs in the radial direction, and hence these spherical waves are longitudinal; the amount of the displacement decreases as the distance from the source of sound increases.

A generalization of the particular solution (230) of the wave-equation (222) is obtained by adding together several such solutions—that is, by using the following expression for the displacement potential:

$$\phi = \frac{1}{r_1} g_1(r_1 - at) + \frac{1}{r_2} g_2(r_2 - at) + \ . \ . \ . \qquad (231)$$

where r_1, r_2 . . . denote the distances of the reference-point from certain fixed centres, and g_1, g_2 . . . denote some functions of a single argument. This motion of the fluid is produced by the combined action of various spheres pulsating in it according to arbitrary laws, if the spheres are so far distant from one another that we may neglect the disturbance which the boundary conditions at the surface of one sphere suffers owing to waves which start out from the other spheres. If, in particular, we take only two spheres and set $g_1 = g_2 = g$, we get:

$$\phi = \frac{1}{r_1} g(r_1 - at) + \frac{1}{r_2} g(r_2 - at) \quad . \quad . \qquad (232)$$

This case may be realized in another interesting way. If we picture in our minds the plane of symmetry of the two centres $r_1 = 0$ and $r_2 = 0$—that is, the plane which bisects the line connecting them perpendicularly—it follows that for every reference-point situated in this plane of symmetry the displacement $q = q_1 + q_2$ occurs within the plane, since the resultant of the two radial displacements, q_1 and q_2, which are equal to each other, bisects their angle (Fig. 9). If we now make the plane of symmetry a rigid wall, the motion is in no wise disturbed, because the boundary condition which holds at the rigid wall—namely, that the displacement component in the direction of its normal should vanish—is already fulfilled everywhere in any case. Thus the motion will take place in exactly the same way if the rigid wall is introduced and the half-space with the sound-source 2 is omitted altogether; in other words, when a rigid plane wall is placed opposite a source of sound it acts exactly like the image of the source with respect to the wall.

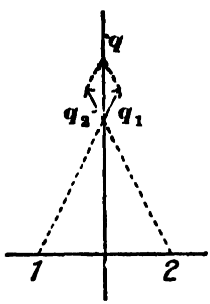

FIG. 9.

Let us next consider a spherical wave that advances inwards, such as is represented in the equation (229) by the function $f(r + at)$. We may imagine it to be realized by a large thin-walled, hollow sphere filled with air and made of some elastic material which is caused to expand and contract by some external mechanism. All the details of the motion are then uniquely determined by the equation (229), combined with the initial and boundary conditions. A particularly interesting question is the following: what happens to a spherical wave which proceeds inwards and is given in some way, when it reaches the centre of the sphere, $r = 0$? To answer this question we shall try the assumption that initially, at the time $t = 0$, only one wave $f(r)$, which is advancing inwards, exists, its form being arbitrary. Since the centre of

the sphere is part of the fluid, the equation (229) also holds for it, and since no infinitely great displacements occur in Nature, the displacement potential is finite also for $r = 0$, and we have for all times:

$$0 = f(at) + g(-at)$$

or, if we write $at - r$ for at:

$$0 = f(at - r) + g(r - at)$$

Substituted in (229) this gives:

$$r\phi = f(at + r) - f(at - r) \quad . \quad . \quad . \quad (233)$$

Exactly as was deduced earlier, in § 37, by means of the equations (168), we find that here also a kind of reflection of the spherical wave occurs at the centre, in that the wave which proceeds inwards becomes transformed into a wave which is opposite and equal to it and advances outwards. We may also express this by saying that the spherical wave passes through itself at the centre of the sphere. There is no contradiction between equation (233) and the convention we made above that only a wave which proceeds inwards is present. For the function $f(r)$ which represents the form of the wave for $t = 0$ is defined only for positive values of r, whereas in (233) also negative values of the argument $at - r$ came into question. Thus we have only to assume the function f to be zero for all negative values of the argument in order to be able to derive the course of the motion uniquely from (233) for our case. For then we actually find for $t = 0$ that:

$$r\phi = f(r)$$

and in the same way for all values of $t < \frac{r}{a}$ the wave which advances outwards vanishes or, what amounts to the same thing, the reference-point r is reached by the reflected wave only when it has traversed the path from the centre of the sphere to it with the velocity a.

§ 48. The theory of the spherical waves also gives us

a solution of the general wave-equation (222) for a fluid which is unbounded on all sides. For if we first integrate this equation over an arbitrary portion of fluid whose element of volume is $d\tau$, we get, taking into account the transformation (82):

$$\int \ddot{\phi} \,.\, d\tau = -a^2 \int \frac{\partial \phi}{\partial \nu} \cdot d\sigma \quad . \quad . \quad . \quad (234)$$

We now choose as our volume of integration a sphere of radius r, the origin being taken as the centre, and we introduce the polar co-ordinates ρ, θ, ψ (I, (92)) as the variables of integration. We then get by I, (93):

$$d\tau = \rho^2 d\rho \,.\, d\Omega \quad . \quad . \quad . \quad . \quad (235)$$

where we have used the abbreviation:

$$\sin\theta \,.\, d\theta \,.\, d\psi = d\Omega \quad . \quad . \quad . \quad . \quad (236)$$

$d\Omega$ is the "angle of aperture" * of the cone which is defined by the differentials $d\theta$ and $d\psi$, and is measured by the size of the area which this cone cuts out of a spherical surface described about the origin as centre and with the radius $\rho = 1$—namely, the "unit sphere."

Further:

$$d\sigma = r^2 \,.\, d\Omega, \quad \nu = -\rho \quad . \quad . \quad . \quad (237)$$

Consequently:

$$\iint \ddot{\phi} \rho^2 d\rho d\Omega = a^2 r^2 \int \left(\frac{\partial \phi}{\partial \rho}\right)_r d\Omega \quad . \quad . \quad (238)$$

The integration over Ω is to be performed on both sides of the equation between the limits $\theta = 0$ to $\theta = \pi$, and $\psi = 0$ to $\psi = 2\pi$, and on the left also over ρ from 0 to r. The index r on the right-hand side is to denote that after the differentiation has been performed we must set $\rho = r$.

We shall now introduce the *mean value* of the displacement potential ϕ for a definite distance ρ from the origin—that is, the quotient of the sum of the values of ϕ—at all the surface-elements of the sphere of radius ρ,

* Solid angle.

each multiplied by the size of the surface-element, by the sum of all the surface-elements—that is, the surface of the whole sphere:

$$\frac{1}{4\pi\rho^2}\int\phi\rho^2 d\Omega = \frac{1}{4\pi}\int\phi d\Omega = \bar{\phi}. \quad . \quad . \quad (239)$$

The above equation may then be written:

$$\int_0^r \frac{\partial^2\bar{\phi}}{\partial t^2}\rho^2 d\rho = a^2 r^2 \left(\frac{\partial\bar{\phi}}{\partial\rho}\right)_r$$

If we differentiate it with respect to r, it follows that:

$$r^2\frac{\partial^2\bar{\phi}}{\partial t^2} = a^2\frac{\partial}{\partial r}\left(r^2\frac{\partial\bar{\phi}}{\partial r}\right) = a^2\left(2r\frac{\partial\bar{\phi}}{\partial r} + r^2\frac{\partial^2\bar{\phi}}{\partial r^2}\right)$$

where we have now to suppose r substituted for ρ in $\bar{\phi}$. But this equation is identical with the wave-equation:

$$\frac{\partial^2(r\bar{\phi})}{\partial t^2} = a^2\frac{\partial^2(r\bar{\phi})}{\partial r^2} \quad . \quad . \quad . \quad . \quad (240)$$

and this enables us to enunciate the theorem: for any arbitrary vibratory process in a fluid the laws of spherical waves hold for the mean values of the displacement potential referred to the spherical surfaces described around any point of the fluid chosen as the origin. For spherical waves we have in particular that $\bar{\phi} = \phi$—that is, (240) becomes identical with (228) in this case.

An interesting consequence of (240) follows if we apply it to the limiting case of an incompressible fluid for which by (215) the velocity of propagation $a = \infty$. For then the wave-equation (222) reduces to $\Delta\phi = 0$—that is, to Laplace's equation I, (129), and the equation (240) becomes $\frac{\partial^2(r\bar{\phi})}{\partial r^2} = 0$, whose general solution is:

$$r\bar{\phi} = A + Br$$

or:

$$\bar{\phi} = \frac{A}{r} + B \quad . \quad . \quad . \quad . \quad . \quad (241)$$

Since $\bar{\phi}$ is finite for $r = 0$, it follows that $A = 0$, and:

$$\bar{\phi} = B \quad . \quad . \quad . \quad . \quad . \quad . \quad (242)$$

that is, a function ϕ, which satisfies Laplace's equation. For example, the potential function of gravitating masses outside these masses has the property that its mean values on a family of concentric spherical surfaces are all equal to one another and hence also equal to the value at the centre. It therefore also follows that the function cannot have an absolute maximum or an absolute minimum at a point in space, and by I, § 41 that there can never be a position of absolutely stable or absolutely unstable equilibrium outside a system of gravitating bodies.

§ 49. The law which is expressed in (240) may now be used to determine the displacement potential at any time at a point of the fluid, which is taken as origin of co-ordinates, for any sufficiently extensive mass of fluid in a given initial state. First the general integral:

$$r\bar{\phi} = f(r + at) + g(r - at)$$

follows from (240) or, since $\bar{\phi}$ remains finite for $r = 0$, we have exactly as in (233) for spherical waves:

$$r\bar{\phi} = f(at + r) - f(at - r) \quad . \quad . \quad . \quad (243)$$

The form of the function f results uniquely from the condition of the initial state. To characterize this state we may assume that for $t = 0$ both the displacement potential ϕ as well as its differential coefficient with respect to the time—namely $\dot{\phi}$—are given as functions of position. For ϕ is determined by the displacements q according to (218), except for an additive constant which is of no consequence, and $\dot{\phi}$ is determined by the velocity $\dot{q}$ according to (219). So we write:

$$\phi_0 = F(r, \theta, \psi) \text{ and } \dot{\phi}_0 = \Phi(r, \theta, \psi) \quad . \quad . \quad (244)$$

and assume F and Φ given. Then:

$$\bar{\phi}_0 = \bar{F}(r) \text{ and } \bar{\dot{\phi}}_0 = \bar{\Phi}(r) \quad . \quad . \quad . \quad (245)$$

are also to be regarded as given functions of r. On the other hand, if we differentiate (243) with respect to t:

$$r\dot{\bar{\phi}} = af'(at + r) - af'(at - r) \quad . \quad . \quad (246)$$

and so for $t = 0$, from (243), (246) and (245):

$$r\bar{F} = f(r) - f(-r) \quad . \quad . \quad . \quad . \quad (247)$$

$$r\bar{\Phi} = af'(r) - af'(-r) \quad . \quad . \quad . \quad . \quad (248)$$

By integrating the last equation:

$$\frac{1}{a}\int r\bar{\Phi}dr = f(r) - f(-r)$$

and hence, combining this with (247), we get:

$$f(\pm r) = \frac{1}{2}\left\{\frac{1}{a}\int r\bar{\Phi}dr \pm r\bar{F}\right\} \quad . \quad . \quad . \quad (249)$$

This determines the function f for all positive and negative values of its argument and hence by (243) $\bar{\phi}$ may be directly given as a function of r and t. The additive constant in the integral may be arbitrarily chosen and has no influence on the value of $\bar{\phi}$.

From the expression of $\bar{\phi}$ for an arbitrary r and an arbitrary t we may also immediately derive the value $\phi_t(0)$ for $r = 0$. For since by (243):

$$\phi_t(0) = \bar{\phi}(0) = \lim_{r=0}\frac{f(at + r) - f(at - r)}{r} = 2f'(at)$$

and since by (249):

$$\pm f'(\pm r) = \frac{1}{2}\left\{\frac{r}{a}\bar{\Phi}(r) \pm \frac{d(r\bar{F}(r))}{dr}\right\}$$

it follows that for all positive values of the time:

$$\phi_t(0) = t\bar{\Phi}(at) + \frac{d(t\bar{F}(at))}{dt} \quad . \quad . \quad . \quad (250)$$

This equation gives us the displacement potential ϕ—at any point of the fluid at any time t, expressed in terms

of the values (244) of ϕ and $\dot{\phi}$ for $t = 0$ on the spherical surface described about the point with the radius at. Among other things, it follows from this that the physical state at any point at any time is dependent only on the physical state at the time $t = 0$ at the distance at from the point. If, for example, the vibrations are caused by a disturbance of the equilibrium which is initially restricted to a limited region of the fluid, the focus of disturbance, then a point which is outside the focus of disturbance will be struck by the disturbing wave only when that time has elapsed which is necessary for the distance to the nearest point of the focus of disturbance to be traversed with the velocity a; on the other hand, the motion becomes fully extinguished when that time has elapsed which is required for the distance from the furthermost point of the disturbing focus to be traversed with the velocity a. The same applies to a point which lies within the disturbing focus. Accordingly, the initial disturbance propagates itself from its focus in all directions with the velocity a, and in the interior a continually expanding region of rest forms which begins at the point for which the distance from the furthermost point of the disturbing focus is least.

The equation (250), besides holding for the displacement potential, also holds immediately for every individual component of the displacement or of the velocity, and likewise for the volume dilatation and the pressure, as indeed for every quantity which satisfies the wave-equation (222).

§ 50. Let us lastly investigate the influence which a *motion* of the source of sound or of the observer has on the pitch of the tone that is heard. The existence of such an influence follows from the fact that the pitch of the tone is conditioned by the number of waves that impinge on the ear per unit of time, and that this number depends on whether the distance of the observer from the source remains unchanged or not. But it would be incorrect to assume that in the question of pitch only the

relative motion of the source and the observer come into consideration. For let us consider the case where the source is at rest and the observer moves away from it with the speed of sound a; he will then hear nothing, because the sound-waves will not reach him at all: whereas, on the other hand, he will certainly be struck by sound-waves if he himself is at rest and the source of sound moves away from him at any arbitrary great speed whatsoever. The reason for this difference is not that the absolute velocity is involved—for there is no meaning in the term "absolute velocity"—but that in the propagation of sound the medium which effects the transmission, such as the air, plays an essential part, and that therefore the motions relative to the air must be taken into account. The laws which come into force

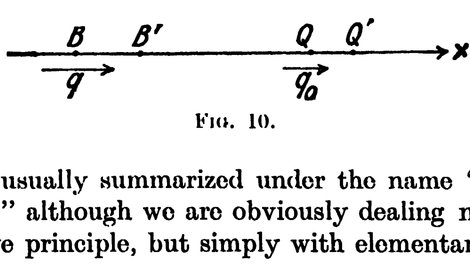

FIG. 10.

here are usually summarized under the name "Doppler principle," although we are obviously dealing not with a distinctive principle, but simply with elementary laws of kinematics.

We assume that a source Q, which emits ν_0 vibrations per unit of time, and an observer B both move along the x-axis, the former with the velocity q_0, the latter with the velocity q, relative to the air, which we take to be at rest. The source is on the positive side (to the right) of B. It then follows that if $q > q_0$ the observer approaches the source (Fig. 10). What is the frequency ν of the tone which is heard by the observer?

Let us first consider the particular case where the source Q is at rest and the observer B approaches it with the velocity q. He then traverses the distances $BB' = q$ in unit time. If the observer had remained at B he would have received ν_0 waves in this time. But since he has

moved towards the waves he has received besides these ν_0 waves all the waves which occupy the distance BB'—that is $\frac{q}{\lambda_0}$—and so, by (189), $\frac{q\nu_0}{a}$ waves. Consequently the frequency of the tone he hears is:

$$\nu_0\left(1 + \frac{q}{a}\right) = \frac{a + q}{\lambda_0} \quad . \quad . \quad . \quad . \quad (251)$$

We shall now assume more generally that the source Q also moves—namely, with the velocity q_0 to the right. After unit time the source then arrives at a point Q', where $QQ' = q_0$. Of the sound-waves which have moved out from the source during this time in the direction of B, the first has in the meantime advanced to within the distance a of Q; the last has arrived at Q', and this whole distance $a + q_0$ is uniformly occupied by waves. Since these are ν_0 waves altogether, the wave-length amounts to $\frac{a + q_0}{\nu_0} = \lambda$, and if this wave-length λ is substituted in (251) in place of λ_0, we get for the frequency of the tone that is heard:

$$\nu = \frac{a + q}{a + q_0} \cdot \nu_0 \quad . \quad . \quad . \quad . \quad (252)$$

This is the general expression of the Doppler principle for the case under consideration. Thus if the observer and the source move at the same speed, so that the distance between them remains constant, then $\nu = \nu_0$ and the motion has no influence at all on the pitch. But if the distance between the observer and the source changes, the result is essentially different according to whether the observer or the source is at rest. If the observer moves away from the source when it is at rest, with the speed of sound, then $q = -a$ and $q_0 = 0$, so that $\nu = 0$. But if the source moves away with the velocity of sound from an observer who is at rest, then $q_0 = a$ and $q = 0$, so that $\nu = \frac{\nu_0}{2}$, which is in accord with our earlier remark.

It is only when q and q_0 are small compared with a that we have to a first degree of approximation :

$$\nu = \left(1 + \frac{q - q_0}{a}\right)\nu_0 . \quad . \quad . \quad . \quad (253)$$

In this case only the *difference* of the velocities—that is, the relative motion of the observer and the source of sound—plays a part.

PART THREE

FINITE DEFORMATIONS

CHAPTER I

GENERAL REMARKS

§ 51. PASSING on from the special case of motions with infinitely small deformations to the more general case of motions with finite deformations, we first again link up with the discussion of an arbitrary finite motion of a continuously extended material body as given in § 2. To represent the latter motion we again use the equations (1) and (1a), from which the position (x, y, z) at which the material point (a, b, c) is situated at the time t is to be calculated.

Conversely, we get from (1), solving in terms of a, b, c, equations of the form:

$$\left.\begin{aligned} a &= f'(x, y, z, t) \\ b &= \phi'(x, y, z, t) \\ c &= \psi'(x, y, z, t) \end{aligned}\right\} \quad . \quad . \quad . \quad (254)$$

which answers the question as to which material point (a, b, c) is situated at the point (x, y, z) at the time t.

The antithesis expressed in equations (1) and (254) is representative of the two points of view that run through the whole of hydrodynamics and give it a dualistic character throughout; these may be called the "substantial" and the "local" points of view. In the former we fix our attention on a definite material point (a, b, c) or a definite material system and then inquire into its changes in space; in the latter view we fix on a definite point in space (x, y, z) or a definite portion of space and inquire into the material points which pass through this point of space or enter into the portion of space in question. The substantial view is characterized by the

choice of a, b, c, t as independent variables, the local view by that of x, y, z, t. To emphasize this difference as much as possible in our nomenclature, we shall in the sequel denote the differentials that refer to the independent variables a, b, c, t by ordinary d's throughout, but those that refer to the independent variables x, y, z, t by ∂'s. Then, for example, $\frac{dx}{dt} = u$ is the x-component of the velocity of the material point (a, b, c), whereas $\frac{\partial x}{\partial t} = 0$. Further, $\frac{du}{dt}$ is the x-component of the acceleration, whereas $\frac{\partial v}{\partial t}$ refers to the difference in the velocities of those two material points which are situated at the point (x, y, z) at the times t and $t + \partial t$. Hence for a stationary efflux of fluid (§ 62) from a vessel we have everywhere $\frac{\partial u}{\partial t} = 0$, but the acceleration $\frac{du}{dt} \neq 0$. In general the following relationship holds between these two quantities:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot u + \frac{\partial u}{\partial y} \cdot v + \frac{\partial u}{\partial z} \cdot w \quad . \quad (255)$$

§ 52. Concerning the purely kinematic aspect of the motions which are to be considered, it is understood that all the laws derived in the first part of the present volume are valid here. Of them we shall in particular use those which relate to an arbitrary infinitely small deformation: they have been formulated in § 12. For any motion may be reduced to an infinitely small change so long as we consider it only during an infinitely small interval of time, from t to $t + \tau$. Then all the formulae of § 12 apply here, with the difference that now the co-ordinates of a material point are no longer called a, b, c before the change, but x, y, z, and that the components of the infinitesimal displacement are no longer u, v, w, but $u\tau$, $v\tau$, $w\tau$, where u, v, w denote the finite components of the velocity. Accordingly, also the nine infinitely

small coefficients λ, μ, ν in (57), which characterize the rotation and the deformation of a mass-particle, become proportional to the element of time τ.

In order now to be able to calculate with finite values, we divide all these infinitely small quantities by τ—that is, we introduce in place of the infinitely small components of the translation, rotation and deformation, the finite components of the *velocity of translation, the velocity of rotation and the velocity of deformation*, and to designate them we again use the same notation as in § 12. Hence from now on:

$$u, v, w\,;\ \xi, \eta, \zeta\,;\ x_x, y_y, z_z, y_z, z_x, x_y \quad . \quad . \quad (256)$$

denote the corresponding finite components of velocity. Relationships hold between them and the co-ordinates x, y, z, which are similar in form to the earlier relationships (59), (60), (61), and so forth.

If we are given the velocity of volume dilatation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div} \boldsymbol{q}$$

of a mass-particle, then we also know its change of density. For we have, exactly as in (213a), for the change of a mass-particle of initial volume dV in the infinitely small interval of time dt:

$$dV \,.\, k = dV\left(1 + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dt\right) . \left(k + \frac{dk}{dt}dt\right)$$

and hence:

$$k\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{dk}{dt} = 0. \quad . \quad . \quad (257)$$

or, in view of the fact that:

$$\frac{dk}{dt} = \frac{\partial k}{\partial x}u + \frac{\partial k}{\partial y}v + \frac{\partial k}{\partial z}w + \frac{\partial k}{\partial t} \quad . \quad . \quad (258)$$

we get:

$$\frac{\partial (ku)}{\partial x} + \frac{\partial (kv)}{\partial y} + \frac{\partial (kw)}{\partial z} + \frac{\partial k}{\partial t} = 0 \quad . \quad . \quad (259)$$

It is interesting to derive this identity, which is often

called the "equation of continuity," by means of a discussion involving a local consideration instead of by one invoking the whole substance. If we fix our attention on any finite space limited in any definite way, then at the time t the total mass contained in it is $\int k d\tau$, and its change in the time dt is :

$$dt\int\frac{\partial k}{\partial t}\cdot d\tau.$$

On the other hand, this change is equal to the algebraic sum of the masses of all those particles of the body which enter into the space through the surface during the time dt. Now the following mass enters through the surface-element $d\sigma$, whose inward normal is ν, in the time dt :

$$d\sigma\,.\,k\,.\Big(u\cos(\nu x)+v\cos(\nu y)+w\cos(\nu z)\Big)\,.\,dt\quad . \qquad (260)$$

This is the mass of an oblique cylinder of density k, of base $d\sigma$ and whose length of side is $q.dt$. Consequently we have the equation :

$$\int\frac{\partial k}{\partial t}\cdot d\tau=\int d\sigma k\Big(u\cos(\nu x)+v\cos(\nu y)+w\cos(\nu z)\Big)$$

or, by (78) :

$$\int\left(\frac{\partial k}{\partial t}+\frac{\partial(ku)}{\partial x}+\frac{\partial(kv)}{\partial y}+\frac{\partial(kw)}{\partial z}\right)d\tau=0$$

This equation also remains valid if we allow the space to contract to a single element of space $d\tau$, from which we then arrive at (259).

Instead of referring the equation of continuity to an infinitesimal element of time dt, we may refer it to a finite time t by setting the mass of an element of the body at the time t equal to the mass of the same element of the body at the time 0, where we use the expression (51) of the functional determinant for the change of volume. Then :

$$dV_0\,.\,k_0=dV_0\,.\,D\,.\,k$$

where the index 0 denotes that we must set $t = 0$. Consequently :

$$D \,.\, k = k_0 \text{ or } \frac{d(Dk)}{dt} = 0 \quad . \quad . \quad . \quad (261)$$

In the sequel we shall use either form of the equation of continuity according to requirements.

§ 53. Our next object is to set up the dynamical equations of motion. If we wish to retain also for arbitrary finite deformations the hypothesis of "perfect elasticity" introduced in § 21, namely that the pressure depends only on the momentary deformation, we are compelled to restrict ourselves to gases and liquids in the following discussion; for in the case of a solid body the elastic limit would be inevitably exceeded sooner or later, as the deformation increases continuously. Hence this third part of the present volume represents the proper domain of hydrodynamics (including aerodynamics).

So, as in § 44, we again set :

$$Y_z = Z_x = X_y = 0 \text{ and } X_x = Y_y = Z_z = p = f(k) \quad . \quad (261a)$$

and we then obtain from (83) the *fundamental equations of hydrodynamics* :

$$\left(X - \frac{d^2x}{dt^2}\right) k - \frac{\partial p}{\partial x} = 0, \quad . \quad . \quad . \quad . \quad (262)$$

These equations may be written still more simply if we assume that the body-force has a potential—that is :

$$X = -\frac{\partial V}{\partial x}, \; Y = -\frac{\partial V}{\partial y}, \; Z = -\frac{\partial V}{\partial z} \quad . \quad (263)$$

and if, in addition, we introduce the function of the pressure or of the density :

$$P = \int \frac{dp}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (264)$$

defined except for an additive constant which remains undetermined.

For then the equations (262) may be written:

$$\frac{d^2x}{dt^2} + \frac{\partial V}{\partial x} + \frac{\partial P}{\partial x} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (265)$$

or in vectorial form:

$$\dot{q} + \text{grad}\,(V + P) = 0 \quad . \quad . \quad . \quad (266)$$

These equations are not, however, as simple as they appear. For in them the co-ordinates x, y, z occur once (in the acceleration) as dependent variables and once (in the potential and pressure gradients) as independent variables, and if we wish to calculate with them, it is necessary as a rule to adopt a uniform point of view—that is, to use either the substantial point of view throughout or the local standpoint. According as we use the one or the other, we obtain different forms of the equations of motion, both of which have a more complicated structure than (265) or (266).

To adopt the substantial point of view we multiply the three equations (265) respectively by $\frac{dx}{da}, \frac{dy}{da}, \frac{dz}{da}$, and obtain:

$$\frac{d^2x}{dt^2}\frac{dx}{da} + \frac{d^2y}{dt^2}\frac{dy}{da} + \frac{d^2z}{dt^2}\frac{dz}{da} + \frac{dV}{da} + \frac{dP}{da} = 0 \,. \quad (267)$$

and, in a similar way, the two equations for b and c.

These are the so-called Lagrange equations. They are supplemented by the equation of continuity in its substantial form (261).

On the other hand, we obtain for the local point of view from (265), using (255):

$$\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t} + \frac{\partial V}{dx} + \frac{\partial P}{\partial x} = 0 \,. \quad (268)$$

and the two corresponding equations for y with v and z with w.

These equations bear Euler's name. The local form (259) of the equation of continuity belongs to them.

As we see, the equations of motion have not only

become longer through each of these methods of formulation, but they have also lost their linear character, which is the principal matter, and it is this circumstance which causes the peculiar mathematical difficulties which present themselves in hydrodynamic problems.

§ 54. The first use that we shall make of the equations of hydrodynamics is to apply and verify the principle of conservation of energy. We shall adopt exactly the same procedure as in § 23 for infinitesimal deformations, by starting out from the energy equation (89) and substituting the values for the quantities that occur in it—namely, the kinetic energy L, the potential energy U, the external work A, after the manner of equations (90), (91) and (92). But now we must take care to note that the components of the displacements of a material point that occur in the time dt are not represented here by du, dv, dw, but by $u\,.\,dt$, $v\,.\,dt$, $w\,.\,dt$. Moreover, it is convenient in this case, where the density is variable to a finite extent, to refer the potential energy not to volume-elements, but to mass-elements. In this way we obtain from (90):

$$L = \frac{1}{2}\int(u^2 + v^2 + w^2)k d\tau \quad . \quad . \quad . \quad (269)$$

and from (91):

$$U = \int F\,.\,k d\tau \quad . \quad . \quad . \quad . \quad . \quad (270)$$

where F denotes the potential energy of unit mass which depends only on k (or p).

And from (92) we get:

$$A = dt\,.\int(Xu + Yv + Zw)k d\tau$$
$$+ dt\,.\int(X_\nu u + Y_\nu v + Z_\nu w)d\sigma \quad . \quad . \quad (271)$$

where, by (74):

$$X_\nu = p\cos(\nu x),\ Y_\nu = p\cos(\nu y),\ Z_\nu = p\cos(\nu z) \ . \quad (272)$$

In forming the time differentials dL and dU we observe that the product $kd\tau$, the mass of an element of the body,

is independent of the time, and so we obtain from (269) that:

$$dL = dt \,.\int\left(u\frac{du}{dt} + v\frac{dv}{dt} + w\frac{dw}{dt}\right)kd\tau$$

and from (270) that:

$$dU = dt \,.\int\frac{dF}{dt}\cdot kd\tau$$

Further, from (271) and (272), if we use the transformation (78), we get:

$$A = dt \,.\int(Xu + Yv + Zw)kd\tau$$

$$- dt \,.\int\left(\frac{\partial(pu)}{\partial x} + \frac{\partial(pv)}{\partial y} + \frac{\partial(pw)}{\partial z}\right)d\tau \quad . \quad (272a)$$

If we substitute these expressions in the energy equation (89) and then take into account the values of the acceleration components according to (262), a number of terms cancel out on both sides of the equation, and we are left with the relation:

$$\int\frac{dF}{dt}kd\tau = -\int p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)d\tau$$

or, since this equation also holds for an individual element of volume $d\tau$, we have, taking (257) into account:

$$dF = \frac{p}{k^2}dk \quad . \quad . \quad . \quad . \quad . \quad (273)$$

Hence the demands of the principle of energy are always and only fulfilled if we take as the potential energy of unit mass:

$$F = \int\frac{p}{k^2}dk = -\int p \,.\, d\left(\frac{1}{k}\right) \quad . \quad . \quad . \quad (274)$$

where $\frac{1}{k}$ denotes the volume of unit mass (specific volume). Instead of this expression we may also write, if we integrate by parts and introduce the function P from (264):

$$F = -\frac{p}{k} + \int\frac{dp}{k} = -\frac{p}{k} + P \quad . \quad . \quad (275)$$

If the potential energy is known as a function of the density or of the specific volume, we can directly calculate from it the dependence of the pressure p on the density. For from (274) it follows that if we denote the specific volume by v for only this case, then :

$$\frac{\partial F}{\partial v} = -p \quad . \quad . \quad . \quad . \quad . \quad (276)$$

This relationship is strikingly analogous to the relationships (97) between the elastic potential and the elastic components of pressure, which are, however, more general in that they also include the shearing pressures, but more special in that they refer only to infinitesimal deformations.

In an *incompressible* fluid we have, in particular, that $k =$ const., and so by (264) :

$$P = \frac{p}{k} \quad . \quad . \quad . \quad . \quad . \quad (277)$$

and by (275) :

$$F = 0$$

That is, an incompressible fluid has no potential energy. This result is in agreement with the conclusion that the incompressibility of a fluid may be regarded, like the inextensibility of a thread (I, § 107), as a condition which has been imposed on the co-ordinates of the particles of the body (volume-dilatation = zero), no matter what the magnitude of the pressures may be, and that the constraints thereby called into action never perform work (I, (314)).

For later use we shall also note that from (271), (272) and (272*a*) we get the following special expression for the work done by an external pressure which acts *uniformly* at all points of the surface during an arbitrary change :

$$A = dt\, p \int (u \cos(\nu x) + v \cos(\nu y) + w \cos(\nu z))d\sigma =$$
$$- dt\, p \int \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) d\tau$$

or, since $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dt$ denotes the dilatation of the element of volume $d\tau$ that occurs in the time dt:

$$A = -pdV \quad . \quad . \quad . \quad . \quad . \quad (278)$$

where dV denotes the change of volume of the whole mass of fluid.

§ 55. Passing on to the special applications of the hydrodynamic laws, we next inquire into the conditions of *equilibrium*. For this we get by integrating (265):

$$V + \int \frac{dp}{k} = \text{const.} \quad . \quad . \quad . \quad (279)$$

Hence in equilibrium the level surfaces, $V = \text{const.}$, of the mass forces (I, § 40) are at the same time surfaces of constant pressure and constant density. If two different fluids border on each other the density k will in general change abruptly at the surface of separation, whereas the pressure p remains continuous in all circumstances, in accordance with the law of action and reaction. Hence if a uniform external pressure acts on the surface of a fluid, the pressure at the boundary of the fluid is also constant, and the surface is therefore a surface of constant potential of the body-forces. In particular this holds for the special case where the external pressure is equal to zero, or equal to that of the atmospheric air, and the fluid has, as we say, a "free" surface—a law which gave "level" surfaces their name.

If the value of the constant of integration in (279) is determined by the conditions at the surface, the value of the pressure in the whole of the interior is then given, quite independently of the way in which the liquid is otherwise bounded, and hence of the size and shape of the vessel in which it is contained. If, for example, the only mass-force that acts on the liquid is its own weight, acting in the direction of the negative z-axis, then by (67):

$$X = 0, \; Y = 0, \; Zkd\tau = -gkd\tau$$

and by (263):

$$V = gz + \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (280)$$

which transforms (279) into:

$$\int \frac{dp}{k} + gz = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (281)$$

and for an *incompressible* fluid:

$$p = \text{const.} - kgz \quad . \quad . \quad . \quad . \quad . \quad (282)$$

Hence the pressure at all points at the same height is the same and increases proportionately to the difference of height as the height decreases. The equation (282) contains the laws which hold in communicating tubes, in machines for raising liquids, such as pumps, in liquid barometers and liquid manometers.

In the case of the mercury barometer, for example, if $z = 0$ is the plane of the free surface, which is subject to the atmospheric pressure p_0, and $z = h$ the height of the liquid which ends in the vacuum, then:

$$\text{for } z = 0, \ p = p_0$$
$$\text{for } z = h, \ p = 0$$

and so by (282):

$$p_0 = kgh \quad . \quad . \quad . \quad . \quad . \quad . \quad (283)$$

The "pressure of one atmosphere" is defined as the value of p_0 for $k = 13{\cdot}596$ grm./cm.^{3}, $g = 980{\cdot}6$ cm.^{3}/sec.^{2}, $h = 76$ cms., so that:

$$p_0 = 1{,}013{,}250 \text{ grm./cm. sec.}^2 \quad . \quad . \quad (284)$$

Instead of using atmospheres, we often prefer to use the corresponding values of h ("millimetres of mercury") to express pressure: these values are given by (283) by dividing the pressure p, measured absolutely, by kg.

§ 56. Let us apply the condition of equilibrium (281) to *compressible* fluids—for example, to a column of air of any height. We then require to know the function $f(k)$ in (211). If we may assume the temperature of the air to be uniform—say 0° C.—then we may write:

$$p = ck \quad . \quad . \quad . \quad . \quad . \quad . \quad (285)$$

where c is determined by the circumstance that for the pressure p_0 of one atmosphere $k = k_0 = 0{\cdot}001293$ [grm./cm.^{3}]—that is:

$$c = \frac{p_0}{k_0} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (286)$$

Hence, performing the integration in (281) we get for $z = 0$ and $p = p_0$:

$$z = \frac{p_0}{gk_0} \cdot \log e \frac{p_0}{p} \quad . \quad . \quad . \quad . \quad . \qquad (287)$$

This is the barometric formula for determining heights when there is isothermal equilibrium, and it enables us to find the height z which corresponds to a definite pressure of air p. By using the numbers given and introducing logs to the base 10, we obtain this formula in the form:

$$z = 18{,}400 \log_{10} \frac{p_0}{p} \text{ metres} \quad . \quad . \quad . \qquad (288)$$

For the ratio $p_0 : p$ it is convenient to express the numerator and the denominator in millimetres of mercury.

In the free atmosphere the condition of uniformity of temperature which we have here assumed is fulfilled only in rare cases, since the equalizing of temperatures between the different layers of air takes place comparatively slowly by heat conduction and is being continually disturbed by air-currents. The formula (285) and the conclusions drawn from it in general become invalid, because the pressure depends not only on the density, but also on the temperature. Nevertheless, it would be wrong to think that uniformity of temperature is a necessary assumption for the application of the theory here developed, in particular for the use of the hypothesis of perfect elasticity (§ 21). For this hypothesis does not require that the temperature should remain constant, but only that the pressure p should depend exclusively on the density k. The temperature may very well alter in the process, but it must again, for its

part, be completely determined by the density. An important case, in which this condition is fulfilled, is realized when every possibility of heat conduction is excluded—that is, when all the changes of volume occur, not "isothermally," at constant temperature, but "adiabatically," so that heat is prevented from passing from one particle of the substance to a neighbouring particle. Isothermal processes correspond to the limiting case of an infinitely great heat conductivity of the substance, adiabatic processes to that of an infinitely small conductivity of the substance. In the latter case, too, the pressure depends exclusively on the density; and hence the equation (211) of perfect elasticity also holds for them, but the function $f(k)$ no longer has the isothermal form (285), but, rather, the adiabatic form :

$$p = c'k^{\gamma} \quad . \quad . \quad . \quad . \quad . \quad (289)$$

where the constant γ has the value 1·405 for air, and c' is again obtained from the equation :

$$c' = \frac{p_0}{k_0^{\gamma}} \quad . \quad . \quad . \quad . \quad . \quad (290)$$

Thus in the case of adiabatic processes the pressure increases more during the compression than in that of isothermal cases, and likewise decreases more during the dilatation; this is due to the fact that the air becomes warmed during adiabatic compression and cooled during adiabatic dilatation.

If we next introduce the formula (289) in (281), then, using a method analogous to that above, we get the following barometric formula for height in the case of adiabatic equilibrium :

$$z = \frac{\gamma}{\gamma - 1} \cdot \frac{p_0}{gk_0} \cdot \left(1 - \left(\frac{p}{p_0}\right)^{\frac{\gamma - 1}{\gamma}}\right) \quad . \quad . \quad (291)$$

or, if we use the numerical values given above :

$$z = 27700\left(1 - \left(\frac{p}{p_0}\right)^{0 \cdot 288}\right) \text{ metres} \quad . \quad . \quad (292)$$

This decrease of pressure with height corresponds with the assumption that in each layer of air the temperature is exactly that which results from adiabatic expansion of the air from 0° C. and atmospheric pressure up to the density of the layer in question.

Adiabatic processes play a more important part in vibration processes than in equilibrium states because in the former case the increases and decreases of temperature due to the alternating compression and dilatation follow in such quick succession that the heat conductivity, which is very small in the case of gases, can be neglected entirely in practice. Hence in calculating the velocity of sound from (215) for gases we may make use of formula (289), which, in combination with (290), gives for the velocity of sound in a gas of density k_0 and pressure p_0:

$$a^2 = \frac{\gamma p_0}{k_0} \quad . \quad . \quad . \quad . \quad . \quad (293)$$

so that for air at 0° C. and atmospheric pressure, we get, by inserting the given numerical values:

$$a = 332 \text{ metres/sec.}$$

which agrees with the results of measurement.

The isothermal compressibility would give, in conjunction with (215):

$$a^2 = \frac{p_0}{k_0} \quad . \quad . \quad . \quad . \quad . \quad (294)$$

and so for air at the same temperature and pressure:

$$a = 280 \text{ metres/sec.}$$

which is too small by a considerable amount.

In liquids which can form drops the difference between adiabatic and isothermal compressibility is only small.

§ 57. We shall now consider another simple example of the application of the fundamental equations of hydrodynamics—namely the case of an *incompressible* fluid which rotates about an axis with *uniform angular velocity*

ω. The motion of all the point-masses is here given at the outset. For if we choose the axis of rotation as our z-axis, we get the position of a point of the fluid a, b, c at the time t from equations (11) by substituting in them :

$$\left.\begin{array}{lll} \alpha_1 = \cos(\omega t), & \alpha_2 = -\sin(\omega t), & \alpha_3 = 0 \\ \beta_1 = \sin(\omega t), & \beta_2 = \cos(\omega t), & \beta_3 = 0 \\ \gamma_1 = 0, & \gamma_2 = 0 & \gamma_3 = 1 \end{array}\right\} \quad . \quad (295)$$

and so :

$$\left.\begin{array}{l} x = a\cos(\omega t) - b\sin(\omega t) \\ y = a\sin(\omega t) + b\cos(\omega t) \\ z = c \end{array}\right\} \quad . \quad . \quad (296)$$

If we differentiate these equations twice "substantially" with respect to the time t we get :

$$\frac{d^2x}{dt^2} = -\omega^2 x, \quad \frac{d^2y}{dt^2} = -\omega^2 y, \quad \frac{d^2z}{dt^2} = 0$$

and by substituting in the equations of motion (265) :

$$-\omega^2 x + \frac{\partial(V+P)}{\partial x} = 0, \quad -\omega^2 y + \frac{\partial(V+P)}{\partial y} = 0, \quad \frac{\partial(V+P)}{\partial z} = 0$$

from which we get by integrating and by using (277) :

$$p = \tfrac{1}{2}k\omega^2\rho^2 - kV + \text{const.} \quad . \quad . \quad . \quad . \quad (297)$$

where ρ denotes the distance of the reference-point (x, y, z) from the axis of rotation. We proceed to apply this equation to two interesting cases.

§ 58. Let a heavy incompressible fluid rotate with constant angular velocity ω in a hollow circular cylinder which has a horizontal base and is open at the top. For example, we may picture the liquid to be contained in a glass and to be stirred round rapidly with a spoon, the spoon being quickly removed finally. To avoid the damping effect of the friction at the wall of the vessel, we may assume that the vessel rotates with the liquid. Then no friction occurs at all during the uniform motion, since the motion takes place without deformation.

From (280) and (297) we here have:

$$p = \tfrac{1}{2}k\omega^2\rho^2 - kgz + \text{const.} \quad . \quad . \quad . \quad (298)$$

By § 55 the form of the free surface of the liquid is determined by $p = \text{const.}$—that is, by:

$$\tfrac{1}{2}\omega^2\rho^2 - gz = \text{const.}$$

This is the equation of a paraboloid of revolution described about the z-axis. If we set $z = z_0$ for $\rho = 0$ we get:

$$z = z_0 + \frac{1}{2}\frac{\omega^2}{g}\rho^2 \quad . \quad . \quad . \quad . \quad . \quad (299)$$

Thus the surface of the liquid is lowest in the middle and increases, as we approach the edge, proportionally to the square of the speed of rotation. The value of z_0, and so the absolute value of the depression in the centre, is determined, independently of the amount of the external pressure, by the volume of the liquid, which is exactly as great in rotation as at rest. Hence if the stationary liquid fills the vessel to the height $z = h$, then the following relationship holds for the volume:

$$\int_0^R \int_0^{2\pi} z \,.\, \rho \, d\rho \, d\phi = R^2\pi \,.\, h$$

where R denotes the radius of the cylinder.

By means of (299) we get from this that:

$$z_0 = h - \frac{\omega^2 R^2}{4g} \quad . \quad . \quad . \quad . \quad . \quad (300)$$

The second term on the right-hand side gives the value of the depression of the level of the liquid at the centre.

The elevation of the edge is the value of $z - h$ for $\rho = R$, as given by (299), so that:

$$z - h = \frac{\omega^2 R^2}{4g} \quad . \quad . \quad . \quad . \quad . \quad (301)$$

and so is exactly as great as the depression of the level at the centre.

We see from (298) that the pressure is variable in a horizontal section (z = const.), and is smallest at the centre and greatest at the edge. This may also be easily observed experimentally if we scatter particles of sand on the bottom of the vessel (assumed at rest), the particles being sufficiently heavy not to share in the rotation. When the liquid begins to rotate, these particles move to the centre of the base of the vessel under the influence of the potential gradient.

§ 59. Let an incompressible liquid, whose individual particles, in obedience to Newton's law, gravitate towards a fixed centre, rotate with constant angular velocity ω about an axis which passes through the centre. We wish to find the form of its free surface. This problem has a peculiar interest for us owing to its relationship with the problem of the flattening of the earth at the poles. We shall therefore also illustrate our example by the conditions that obtain on the earth.

If r denotes the distance of the reference point from the centre O, then the gravitational potential is, by I, (111):

$$V = -\frac{c}{r}$$

and by (67) and (263) the attractive force per unit mass (acceleration due to gravitation) is:

$$\frac{\partial V}{\partial r} = \frac{c}{r^2}$$

To determine the constant c we assume that the acceleration due to gravity is equal to g_0 at the distance r_0. We then have:

$$g_0 = \frac{c}{r_0^2}$$

and, generally:

$$V = -\frac{g_0 r_0^2}{r} \quad . \quad . \quad . \quad . \quad . \quad (302)$$

By (297), this gives the following equation for the form of the free surface :

$$\frac{1}{2}\omega^2\rho^2 + \frac{g_0 r_0^2}{r} = \text{const.}$$

or, if we set $\rho = 0$, $r = r_0$ (radius vector) at the surface :

$$\frac{1}{2}\omega^2\rho^2 = g_0 r_0^2\left(\frac{1}{r_0} - \frac{1}{r}\right). \quad . \quad . \quad . \quad (303)$$

If we introduce instead of ρ the angle ϕ (geographical latitude) by means of the relationship :

$$\rho = r\cos\phi$$

then (303) becomes :

$$\frac{1}{r} = \frac{1}{r_0} - \frac{\omega^2}{2g_0 r_0^2}\cdot r^2\cos^2\phi \quad . \quad . \quad . \quad (304)$$

For the special case $\omega = 0$ (liquid at rest) r is constant, so that the surface is that of a sphere. Hence for small values of ω the surface departs only little from that of a sphere and is a spheroid which is flattened at the poles : the equation of the spheroid can be written to a first degree of approximation as :

$$r = r_0\left(1 + \frac{\omega^2 r_0}{2g_0}\cos^2\phi\right) \quad . \quad . \quad . \quad (305)$$

The amount of the flattening is :

$$\frac{r_{\max} - r_0}{r_0} = \frac{\omega^2 r_0}{2g_0} \quad . \quad . \quad . \quad . \quad (306)$$

For the earth, for which :

$$\omega = \frac{2\pi}{24\cdot 60\cdot 60}[\text{sec}^{-1}],\ r_0 = 6{\cdot}356\cdot 10^8[\text{cms.}],\ g_0 = 983\left[\frac{\text{cm.}}{\text{sec.}^2}\right]$$

the flattening comes out as $\frac{1}{585}$; the actual measured value is not inconsiderably greater, being $\frac{1}{298}$.

The difference is easily accounted for by the fact that the mass constituents of the earth do not gravitate

towards the centre of the earth, but towards each other, which clearly favours the departure from the spherical shape. The introduction of mutual gravitation of course makes the problem much more complicated because the expression for the gravitational potential V is then no longer given at the outset, but itself depends on the form of the surface which is to be determined. Closer investigation has shown that the problem when formulated in this way has no unique solution—that is, very different forms of the surface are possible; among them we also find an ellipsoid of rotation with a definite amount of flattening.

If we restrict our attention to small values of the angular velocity, that is, to slight deviation from the spherical shape—then we may set the attractive force on a particle of liquid, to a certain degree of approximation, by I, (102), directly proportional to the distance r from the earth's centre—that is, we set the potential proportional to r^2. If, making this assumption, we carry out the calculation exactly according to the above method, we again arrive at the expression (306) for the amount of the flattening; this is clearly to be ascribed to the circumstance that for small deviations from the spherical shape the form of the law according to which V depends on r does not come into consideration to any marked extent for the flattening.

§ 60. We shall now again resume the general treatment of the theory and shall first perform an important integration of the hydrodynamic equations which was first discovered by Helmholtz. It has an importance in hydrodynamics similar to the principle of areas (law of sectional areas) for general mechanics.

If we start out from the three Lagrange equations (267), which correspond to the three letters a, b, c, the function V and P may obviously be eliminated from them by differentiating the a-equation—say, with respect to b, and the b-equation with respect to a—and subtracting the resulting equations from each other. If we carry

out this operation for each pair of letters we again get three equations which correspond to the letters a, b, c. We shall do this here for the a-equation. This runs :

$$\frac{d}{db}\left(\frac{d^2x}{dt^2}\frac{dx}{dc}+\frac{d^2y}{dt^2}\frac{dy}{dc}+\frac{d^2z}{dt^2}\frac{dz}{dc}\right)$$

$$-\frac{d}{dc}\left(\frac{d^2x}{dt^2}\frac{dx}{db}+\frac{d^2y}{dt^2}\frac{dy}{db}+\frac{d^2z}{dt^2}\frac{dz}{db}\right)=0$$

and after the differentiations have been performed :

$$\sum_{x,y,z}\frac{dx}{dc}\frac{d}{db}\left(\frac{d^2x}{dt^2}\right)-\frac{dx}{db}\frac{d}{dc}\left(\frac{d^2x}{dt^2}\right)=0$$

where the summation sign denotes that a y-term and a z-term have yet to be added to the x-term that has been written down. This expression may be written as a differential coefficient with respect to the time, namely :

$$\sum_{x,y,z}\frac{d}{dt}\left(\frac{dx}{dc}\frac{du}{db}-\frac{dx}{db}\frac{du}{dc}\right)=0$$

as we may see by performing the differentiation with respect to t, since two of the resulting four terms cancel out.

Hence by integrating we obtain :

$$\sum_{x,y,z}\frac{dx}{dc}\frac{du}{db}-\frac{dx}{db}\frac{du}{dc}=A \quad . \quad . \quad . \quad . \quad (307a)$$

where the quantity A depends only on a, b, c and not on t. In precisely the same way we obtain the b-expression $(307b) = B$, and the c-expression $(307c) = C$, where B and C have the same property as A.

To transform the three equations which refer to a, b, c into three equations which refer to x, y, z we multiply the three equations (307) in turn by $\frac{dx}{da}$, $\frac{dx}{db}$, $\frac{dx}{dc}$ and add. By appropriating grouping terms, some of which cancel and some of which may be represented by the symbol introduced in (8), we get :

$$\sum_{a,b,c}\frac{dw}{da}\left[\frac{dy}{da}\right]-\frac{dv}{da}\left[\frac{dz}{da}\right]=A\frac{dx}{da}+B\frac{dx}{db}+C\frac{dx}{dc} \quad . \quad (308x)$$

and, correspondingly, the two equations ($308y$) and ($308z$).

Now, by (53):

$$\sum_{a,b,c} \frac{dw}{da}\left[\frac{dy}{da}\right] = D\frac{\partial w}{\partial y}$$

where D denotes the functional determinant (51). Hence ($308x$) becomes:

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{1}{D}\left(A\frac{dx}{da} + B\frac{dx}{db} + C\frac{dx}{dc}\right)$$

and if we take into account (59) and (256), and also (261) we get:

$$\xi = \frac{k}{2k_0}\left(A\frac{dx}{da} + B\frac{dx}{db} + C\frac{dx}{dc}\right) \quad . \quad . \quad (309x)$$

The equations ($309y$) and ($309z$) for η and ζ are analogous.

To determine the values of the integration constants A, B, C, which are independent of t, we set $t = 0$ in the three equations (309). We then get, since, by ($1a$), $\left(\frac{dx}{da}\right)_0 = 1$, $\left(\frac{dx}{db}\right)_0 = 0$, $\left(\frac{dx}{dc}\right)_0 = 0$, and so forth, that:

$$\xi_0 = \tfrac{1}{2}A, \ \eta_0 = \tfrac{1}{2}B, \ \xi_0 = \tfrac{1}{2}C.$$

Therefore we get generally that:

$$\left.\begin{aligned}
\xi &= \frac{k}{k_0}\left(\xi_0\frac{dx}{da} + \eta_0\frac{dx}{db} + \zeta_0\frac{dx}{dc}\right)\\
\eta &= \frac{k}{k_0}\left(\xi_0\frac{dy}{da} + \eta_0\frac{dy}{db} + \zeta_0\frac{dy}{dc}\right)\\
\zeta &= \frac{k}{k_0}\left(\xi_0\frac{dz}{da} + \eta_0\frac{dz}{db} + \zeta_0\frac{dz}{dc}\right)
\end{aligned}\right\} \quad . \quad . \quad (310)$$

These equations involve a number of consequences which all arise from the fact that the quantities k_0, ξ_0, η_0, ζ_0 depend only on a, b, c and not on t.

Let us first consider a particle of liquid a, b, c whose velocity of rotation at the time $t = 0$ is equal to zero.

Then the components ξ_0, η_0, ζ_0, that refer to it vanish, and it follows from (310) that:

$$\xi = 0, \ \eta = 0, \ \zeta = 0$$

That is, *a particle of fluid whose motion is irrotational at any moment of time retains this property for all time.*

If we further consider for the time $t = 0$, for which the quantities a, b, c of course coincide with the quantities x, y, z, such particles of fluid a, b, c for which the components of the rotational velocity all or partly differ from zero, we can form a picture of the spatial distribution of this vector (ξ_0, η_0, ζ_0) by imagining the following family of curves to be constructed:

$$da : db : dc = \xi_0 : \eta_0 : \zeta_0 \quad . \quad . \quad . \quad (311)$$

Each of these curves has the property that the tangent at any point coincides with the direction of the axis of rotation at this point. Hence such a line is called a *vortex*-line of the fluid. We shall now keep our attention on those material points a, b, c, which, for $t = 0$, belong to a vortex-line and so are connected with each other by the equation (311) and shall follow these points at later times t and inquire into their velocities of rotation. Their magnitude and direction are given by (310). Concerning their direction, we get by substituting the values of ξ_0, η_0, ζ_0 from (311):

$$\xi : \eta : \zeta = dx : dy : dz \quad . \quad . \quad . \quad . \quad (312)$$

That is, the curve which the material points under consideration form in space again has the property of a vortex-line; or *the vortex-lines always consist of the same material points.*

Hence each separate vortex-line forms an individual substantial configuration which, in the course of time, changes its position and its form, but never its material composition and its fundamental property as characterized by (312). A vortex-line may run back into itself or can pass to infinity or to the surface of the liquid.

The equations (310) also give information about the magnitude ω of the rotational or the vortical velocity.

For if we consider an element of length $ds_0 = \sqrt{da^2 + db^2 + dc^2}$ of a vortex-line, then by (311):

$$\frac{\xi_0}{\omega_0} = \frac{da}{ds_0}, \quad \frac{\eta_0}{\omega_0} = \frac{db}{ds_0}, \quad \frac{\zeta_0}{\omega_0} = \frac{dc}{ds_0}$$

and, by substituting in (310):

$$\xi = \frac{k}{k_0}\omega_0 \frac{dx}{ds_0}, \quad \eta = \frac{k}{k_0}\omega_0 \frac{dy}{ds_0}, \quad \zeta = \frac{k}{k_0}\omega_0 \frac{dz}{ds_0}$$

Consequently the rotational velocity at the time t is:

$$\omega = \sqrt{\xi^2 + \eta^2 + \zeta^2} = \frac{k}{k_0}\omega_0 \cdot \frac{ds}{ds_0} \quad . \quad . \quad (313)$$

That is, the rotational velocity at any point of a vortex-line is proportional to the density of the fluid and the length of an element of arc of the line provided that the element of arc always comprises the same material points. For example, if the element of arc becomes lengthened in the course of time owing to the material points that form it moving apart, while k remains unchanged, the vortical velocity increases.

This law assumes a still more striking aspect if we express it in a somewhat different way. Let us take any arbitrary area in the fluid and draw a vortex-line through each of the points on its edge, then a space becomes delimited in the fluid which is called a "vortex-filament" or a "vortex-tube." The enveloping surface of the tube is formed completely of vortex-lines, whereas the surface area taken above represents a cross-section of the tube. We may imagine the whole liquid volume as composed entirely of infinitely thin vortex-filaments which either return into themselves or run off to infinity or end in the surface of the liquid. Like the vortex-lines, so of course the vortex-threads consist only of the same material points.

Let us now consider an infinitely short piece of such an infinitely thin vortex-filament by taking two points

of a vortex-line which belongs to the vortex-filament, the two points being separated by the distance ds; through these two points we draw two cross-sections of area of surface f in any direction (so long as they are parallel). We fix our attention on the material points that belong to this piece of vortex-filament at different times. Since the mass of the piece of vortex-filament is invariable, we then have:

$$k \,.\, f \,.\, ds \,.\, \cos\theta = k_0 f_0 ds_0 \cos\theta_0$$

where we use θ to denote the acute angle which the normal to the cross-sections form with ds—that is, with the axes of the vortex. Then (313) becomes:

$$\omega \,.\, f \,.\, \cos\theta = \omega_0 f_0 \cos\theta_0 = \omega \,.\, f_n \quad . \quad . \quad (314)$$

where f_n denotes the surface of the cross-section normal to the axis of the vortex. *The product of the vortical velocity and the normal cross-section of a vortical filament does not, therefore, change with time.*

Moreover, this product has the same value at different points of a definite vortex-filament. For if we integrate the identity:

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0 \quad . \quad . \quad . \quad . \quad (315)$$

which follows from (59), over any arbitrary volume of liquid whatsoever and use (78) to transform each of the three space-integrals that result in the process, we obtain for the integral that is to be taken over the surface of the space:

$$\int\Big(\xi\cos(\nu x) + \eta\cos(\nu y) + \zeta\cos(\nu z)\Big)d\sigma =$$
$$\int\omega\cos(\nu, \omega) \,.\, d\sigma = 0 \quad . \quad . \quad (316)$$

Now if the volume of fluid is an arbitrarily long section of a vortex-filament, then all those terms vanish in this surface-integral which relate to the enveloping surface of the filament, because at every point of this surface the normal ν to it is perpendicular to the direction of the vortical axis which lies in this surface. Hence we are left with only the terms that refer to the two cross-

sections, and if the vortical filament is infinitely thin but of finite length, with its initial cross-section f and also its final cross-section f' arbitrarily directed, the equation (316) reduces to the two terms:

$$\omega \cos(\nu, \omega) \,.\, f + \omega' \cos(\nu', \omega') \,.\, f' = 0$$

or, if we again use θ to denote the acute angle between the normals to the cross-sections and the axis of the vortex, and observe that the vortex axes ω, ω' point in the same direction but the inward normals ν, ν' in opposite directions, then:

$$\omega f \cos\theta - \omega' f' \cos\theta' = 0 \quad . \quad . \quad . \quad (317)$$

which when combined with (314) leads to the theorem: *in an infinitely thin vortex-filament the product of the vortical velocity and the normal cross-section has the same value at all points of the filament and at all times.* This product is often called the "moment" of the vortex-filament.

The laws of vortex motion deduced above are of course contingent on the assumption that we are dealing with perfectly elastic fluids (§ 53) and a force-potential (263). In Nature we always find more or less considerable deviations from this assumption, with the result that, contrary to these laws, vortices can arise and also disappear. Such deviations are caused in particular by friction (viscosity) and heat conduction. Friction gives rise to pressures in the interior of the fluid; these pressures do not depend on the momentary deformation, but on the momentary rate of deformation (§ 78); heat conduction has the effect that the pressure no longer depends on the density alone and hence the equation (211) is no longer fulfilled. It is only in the limiting cases of infinitely great heat-conduction (isothermal processes) and infinitely small heat-conduction (adiabatic processes) that we may regard the liquid as perfectly elastic, as we saw in § 56, and hence may then assume the laws of vortices to apply when the body-forces are conservative.

CHAPTER II

IRROTATIONAL MOTIONS

§ 61. Now that we have become acquainted with the fundamental importance of vortex motions by integrating the general equations of hydrodynamics, we classify fluid motions in our following treatment under the heads of irrotational (vortex-free) and vortical motions. Let us first consider the former. It is characterized by the condition that:

$$\operatorname{curl} \mathbf{q} = 0 \quad . \quad . \quad . \quad . \quad . \quad (318)$$

or by:

$$\mathbf{q} = -\operatorname{grad} \phi \quad . \quad . \quad . \quad . \quad . \quad (319)$$

That is, a function ϕ, the "velocity potential" exists, whose partial derivatives with respect to x, y, z give us the respective components of the velocity (cf. (217) and (218) above). From the preceding discussion we know that if the equation (318) or (319) holds for any moment of time it also holds for all other times.

The velocity potential ϕ is usually defined with the contrary sign to that which we are using, in that the plus sign is used in (319). But if we wish to maintain the analogy with the force potential and the elastic potential, as well as with the electric and thermodynamic potential, we must also assume in the present case of the velocity potential that the direction of the vector $\mathbf{q}$ is opposite to that of the corresponding gradient. The velocity, like the force in the case of the force potential, is then directed in the sense of *decreasing* potential. In the expression for the velocity potential there is, by (319), an additive time-function which is fully indeterminate and hence of no significance physically.

We obtain a good survey of the velocity state of the liquid at any moment t by drawing the surfaces of constant velocity potential, ϕ = const., and the curves that cut them perpendicularly:

$$dx : dy : dz = \frac{\partial \phi}{\partial x} : \frac{\partial \phi}{\partial y} : \frac{\partial \phi}{\partial z} \quad . \quad . \quad . \quad . \quad (320)$$

namely, the "stream-lines" or "lines of flow," which specify the direction of the velocity at each of its points, in complete analogy with the level surfaces and lines of force (I, § 40). Since a stream-line always passes from a higher to a lower velocity potential, it cannot return into itself if the velocity potential is uniform and continuous. We shall, however, later meet with irrotational fluid motions with closed stream-lines, and shall then have to draw the inference that a definite irrotational motion can also have a many-valued or a discontinuous velocity potential.

From the idea of the stream-line we also immediately derive that of a "stream-filament" or a "stream-tube," which is characterized by having its enveloping surface formed of stream-lines. A stream-filament stands in the same relation to the stream-line as the vortex-filament to the vortex-line. The surfaces of constant velocity potential are of course orthogonal to the stream-filaments.

If the fluid motion is not stationary, the system of stream-lines and stream-filaments will be different at different times—that is, the stream-lines will change with the time. But we must be careful to observe that there is no sense in speaking of the motion of a definite stream-line. For the points of the fluid, which form a stream-line at a certain moment of time, will no longer do so at another moment. Hence in general it is not possible to associate a stream-line at one moment of time with a stream-line at another moment of time. In this respect there is a fundamental difference between stream-lines and vortex-lines, which are always formed

by the same points of the fluid and therefore possess an individual character.

Since the condition of irrotational motion can be expressed most conveniently by means of the space-co-ordinates x, y, z as independent variables, the equations of motion in Euler's form (268) are found to be best for representing irrotational motions. Using (319) we may write them in the form :

$$\frac{\partial u}{\partial x}u + \frac{\partial v}{\partial x}v + \frac{\partial w}{\partial x}w - \frac{\partial^2 \phi}{\partial x \partial t} + \frac{\partial V}{\partial x} + \frac{\partial P}{\partial x} = 0$$

and so forth. Integrating them with respect to x, y, z we get :

$$\frac{u^2 + v^2 + w^2}{2} + V + P - \frac{\partial \phi}{\partial t} = f(t)$$

or, since, as we have already remarked above, an additive constant is arbitrary in the value of the velocity potential, we have :

$$\frac{u^2 + v^2 + w^2}{2} + V + P - \frac{\partial \phi}{\partial t} = 0 \quad . \quad . \quad (321)$$

In addition to this equation, we also have the equation of continuity (259) and the relationship between pressure and density, which depends on the nature of the fluid. We then have three equations from which the three quantities ϕ, p, k may be determined with the help of the initial and boundary conditions corresponding to the particular case; these conditions are expressed as functions of the independent variables x, y, z, t.

§ 62. **Stationary motion of an incompressible fluid.** If the motion is *stationary* the quantities $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$, $\frac{\partial w}{\partial t}$ all $= 0$, and by integrating these equations with respect to x, y, z, we get :

$$\frac{\partial \phi}{\partial t} = \text{const.}$$

If, further, the fluid is incompressible and heavy, P

assumes the value (277) and V the value (280), so that the equation (321) becomes

$$p = -\frac{k}{2}(u^2 + v^2 + w^2) - kgz + \text{const.} \quad . \quad (322)$$

whereas the equation of continuity (259) reduces to:

$$\Delta\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (323)$$

This is the well-known Laplace equation (I, (129)). *Every* solution of this differential equation which is independent of the time represents an irrotational stationary motion which is possible in Nature in an incompressible fluid, the pressure p being determined by (322). As we see, this pressure is composed of two parts, the first of which is often called the hydrodynamic pressure, the second, which is identical with (282), the hydrostatic pressure. The hydrodynamic pressure depends only on the magnitude of the velocity and varies in the reverse sense with it. But neither the pressure itself nor the pressure gradient has anything to do with the direction of the motion of the fluid; we shall encounter some noteworthy examples of this in the next section.

If we fix on a definite and definitely delimited space within the fluid and integrate the equation of continuity (232) over this space, we get by (82):

$$\int \frac{\partial \phi}{\partial \nu} d\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (324)$$

This equation acquires a striking meaning if we reflect that:

$$-k\frac{\partial \phi}{\partial \nu} d\sigma dt = kq_\nu d\sigma dt \quad . \quad . \quad . \quad . \quad (325)$$

represents that quantity of fluid which in the time dt flows through the surface-element $d\sigma$ into the interior of the space. For it states that in the space in question the quantity of fluid that enters as a whole through *all* the surface-elements is zero, or that just as much fluid flows in as flows out.

If this space is an arbitrarily long portion of a stream-tube of arbitrarily great cross-section, then those parts of the surface-integral (324) which refer to the enveloping surface of the tube drop out and the equation reduces to the theorem that in a stream-tube the same quantity of fluid flows through every cross-section. This quantity is therefore characteristic of the stream-tube in question, and, referred to unit time, it is called the "stream-intensity" or "stream-strength" of the tube :

$$J = -k\int\frac{\partial\phi}{\partial\nu}d\sigma \quad . \quad . \quad . \quad . \quad (326)$$

where the integral is now to be taken over any arbitrarily formed cross-section of the tube and the direction of ν must be chosen in the sense of the flow.

If we make the cross-section coincide with a surface ϕ = const., the direction of ν becomes that of the velocity q and we get :

$$J = k\int q d\sigma \quad . \quad . \quad . \quad . \quad . \quad (327)$$

§ 63. In any arbitrary stream-tube we may regard the enveloping surface replaced by a fixed wall without in any way disturbing the motion of the fluid by so doing; for the boundary condition which holds at the fixed wall—namely, that the normal component of the velocity vanishes—is always fulfilled at the enveloping surface of the stream-tube. This endows the stream-tubes with an immediate practical importance. If the cross-section is not too great, and if the velocity of flow is not too irregular in magnitude and direction at the different points of a cross-section, as happens, for example, in the case of water pipes, we may assume to a certain degree of approximation that the surfaces of constant velocity potential are plane, and so place q in front of the integral sign in (327). Accordingly :

$$J = k\,.\,q\,.\,f: \quad . \quad . \quad . \quad . \quad (328)$$

where f denotes the normal cross-section of the tube.

Since J and k are constant along the whole length of the tube, the magnitude of the velocity is inversely proportional to the cross-section f of the tube, or :

$$q = \frac{f_0 q_0}{f} \quad . \quad . \quad . \quad . \quad . \quad (329)$$

where we have designated any definite cross-section, for example, the initial cross-section, by the index zero. From this we then also get, by (322) the pressure at any cross-section f of the tube :

$$p - p_0 = -\frac{k}{2} q_0^2 \left(\frac{f_0^2}{f^2} - 1\right) + kg(z_0 - z) \quad . \quad (330)$$

As we see, the pressure at the narrowest points of the tube has the smallest value, and in principle we can, for a definite q_0 and f_0, bring about any arbitrary decrease of pressure, corresponding to the increase of velocity, by appropriately narrowing the tube—that is, by appropriately decreasing f. If we bore a fine hole in the wall of the tube at a narrow portion of this kind and so effect communication with the outer atmosphere, assumed at normal pressure p_0, then the difference of pressure $p_0 - p$ drives the air into the tube, so that it is dragged along by the fluid. This fact is made use of in hydraulic air pumps and other devices. The action of spray-diffusers also depends on the decrease of pressure with increasing velocity.

The term, in (330), which is due to gravity, may also be used to diminish the pressure p, as in Bunsen's air-pump (cf. filter-pump), in which water falls down a vertical cylindrical tube ($f = f_0$), so that the difference of pressure at two points is simply proportional to the difference in height, $z_0 - z$.

The equations (329) and (330) may also be used conversely to find the stationary velocities q_0 and q, with which the liquid flows through the tube when there is a given difference of pressure $p_0 - p$ at the ends of the tube. For example, let us calculate the stationary velocity of efflux out of a tube which is wide above and

narrow below, by assuming that the upper level of the liquid is maintained at the pressure p_0, but the orifice of escape is kept at the pressure p. If we neglect f in comparison with f_0 and denote the difference in the height of the two levels by h, then we obtain from (330) and (329) that:

$$q^2 = 2gh + \frac{2}{k}(p_0 - p) \quad . \quad . \quad . \quad . \quad (331)$$

for the velocity of efflux.

If the pressure is equal above and below, then $q = \sqrt{2gh}$ —that is, the velocity of efflux is equal to the velocity of a body which has fallen freely through the height h (Torricelli's Theorem). The efflux is retarded or accelerated according to the sign of the pressure-difference. In order that q may remain real, the difference of pressure must not, however, fall below the value $-hkg$. In the limiting case the stationary velocity of efflux becomes equal to zero, and we obtain the well-known barometric formula (283) for the difference of pressure.

In reality the quantity of fluid that escapes is always considerably less (by about a third) than that which is calculated from the velocity of efflux (331) and the magnitude of the orifice of escape, as given by (328). This is essentially due to the fact that the surface ϕ = const. in the orifice of escape is by no means plane, but rather, when viewed from outside, concave, since the stream-lines, as they come from the interior of the liquid, crowd together towards the narrow orifice of escape, like the generators of a cone towards its apex. Hence the surfaces ϕ = const. here behave similarly to the spherical surfaces which surround the apex concentrically. The consequence is that the velocity of flow at the different points of the orifice of escape is not uniform in direction, as was assumed in applying equation (328), but that the stream-lines converge and so the cross-section of the stream-tube contracts somewhat *vena contracta* in front of the orifice of escape, at the point where the fluid already forms a free ray (§ 67). We get better agreement with

reality if, in equation (328), we substitute for f not the size of the orifice of escape, but the smaller cross-section which the ray of fluid has after the contraction has occurred, because there the stream-lines run more parallel.

The amount that flows out may be considerably increased by choosing an appropriate cylindrical or, still better, conical additional tube fixed over the outside of the orifice of escape. This is clearly tantamount to making the stream-lines parallel or divergent, as the case may be, when the fluid flows out and so avoiding the contraction of the ray. Further details of the flow in a conical tube will be found in the next section.

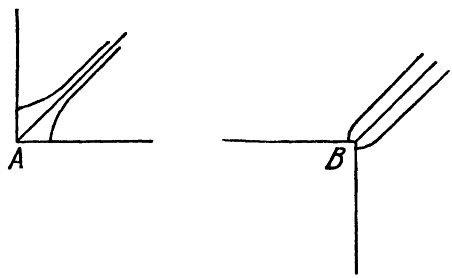

FIG. 11.

If the wall of the tube is not uniformly curved everywhere, but forms an angle at one point, then the streamline which traverses the wall and passes through the apex of the angle has a singularity at this point, since the direction of the line becomes two-fold here. Thus the components of the velocity of flow are either zero or infinite at such an angle. The following consideration will indicate which of these two cases actually occurs. If the wall, as viewed from the liquid, forms a concave angle ($<\pi$), as at the point A in Fig. 11, the surfaces ϕ = const. which fall perpendicularly on the wall and are designated by lines in the figure follow a course such that in approaching A they diverge from the singular potential-surface which passes through A. Then the gradient of ϕ, and with it the value of the velocity at

the point A, becomes equal to zero at the point A (cf. I, § 40); consequently, by (332), the pressure p becomes a maximum. But if the wall forms a convex angle ($>\pi$)—that is, one which projects into the fluid, as at the point B in Fig. 11—the surfaces of constant velocity potential converge towards the singular point, so that the gradient of ϕ and the value of the velocity become ∞ there, and consequently the pressure p becomes equal to $-\infty$, and the fluid would have to have an infinitely great power of cohesion if it is not to be torn asunder.

§ 64. We now turn our attention to a few simple particular solutions of the potential equation (323) and of the motions in the fluid that correspond to them. The simplest solution is that for which ϕ depends linearly on x, y, z. The motion corresponding to it is one of uniform translation of the fluid, which is of no further interest here.

A more general solution is (by I, (125) and (129)):

$$\phi = \sum \frac{\mu_1}{r_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (332)$$

where r_1 denotes the distance of the reference point from a fixed centre 1, and μ_1 a constant which is a characteristic for this centre, and the summation is to be performed over an arbitrarily great number of different centres. Since the velocity potential becomes infinitely great when the reference point approaches infinitely near a centre which has been fixed in any way, the fixed centres themselves must be imagined excluded from the volume of the fluid in some way, which can be accomplished by means of arbitrarily small surfaces which closely surround the individual centres.

Let us next consider the special case of a single centre :

$$\phi = \frac{\mu}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (333)$$

The surfaces of constant velocity potential are concentric spherical surfaces, and so the stream-lines are the

generators of the rectilinear orthogonal beam of rays. If μ is positive, the fluid flows from the centre in all directions and we have a "source" there. If μ is negative, the fluid is drawn into the centre from all directions and we have a "sink" there. The value of the velocity, taken in the direction drawn from the centre outwards, is:

$$-\frac{\partial \phi}{\partial r}=\frac{\mu}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad (334)$$

Thus it varies in inverse proportion to the square of the distance and becomes equal to $\pm\infty$ at the centre itself. Consequently, by (322), the pressure p is always equal to $-\infty$ at the centre, no matter whether we are dealing with a source or a sink. Here we see a noteworthy difference between flow of fluids and stationary galvanic or thermal currents, which start out from a point-electrode or a point-source of heat and which may also be represented by Laplace's differential equation. For in the latter two cases the direction of the flow depends on the potential gradient and the temperature gradient and changes its sign with them; in our case the pressure always increases outwards, no matter whether the liquid is flowing inwards or outwards. The reason for this difference in behaviour is to be found in the inertia of the fluid. For the pressure-gradient here always leads to an inwardly directed force which is necessary, in the case of the source, for gradually retarding the velocity of the fluid which flows outwards, and, in the case of the sink, for gradually increasing the velocity of the fluid that flows inwards.

Any arbitrary conical surface which has its apex at the centre can be imagined to be replaced by a fixed wall; in this way we can obtain certain laws, including the law of efflux through a conically shaped tube fixed over the orifice, which was treated in the previous section. The constant μ is intimately connected with the productivity or rate of supply of the source. If we imagine a definite surface of arbitrary form to be placed in the

liquid, such that it entirely encloses the source, then, on account of the stationary state, the quantity of fluid which flows out as a whole through the surface per unit of time is equal to the quantity supplied by the source in the same time, and hence is entirely independent of the form of the surface; we call it the "intensity" J or the "strength" of the source. Its value can be calculated most simply if we give the imaginary surface the form of a sphere described about the source as centre. We then obtain for the intensity of the source by (325) and (334):

$$J = -k\int\frac{\partial\phi}{\partial r}d\sigma = k\frac{\mu}{r^2}\int d\sigma = 4\pi k\mu \quad . \quad . \quad (335)$$

This equation at the same time represents a generalization of the relationship (324), which holds only when the space enclosed by the surface contains no singular point.

If we now proceed to consider the general case (332) of an arbitrarily great number of sources and sinks, we find precisely similar conditions in their case. The stream-lines begin at the sources and end either at the sinks or run off to infinity. The decisive factor for the productivity of an individual singular point is its constant μ, in that, by (335), it gives the intensity J of the source or the sink. For in the immediate neighbourhood of the point the influence of all the other sources may be neglected, and accordingly the surfaces of constant velocity potential close to the source are the concentric spherical surfaces that surround the source.

If we imagine a certain volume in the fluid to be marked off by some closed surface, then, on account of the stationary state, the quantity of fluid that flows out as a whole through the surface in unit time is equal to the algebraic sum of the quantities delivered by the sources and sinks in the same time, or, by (325) and (335):

$$k\int\frac{\partial\phi}{\partial\nu}d\sigma = \Sigma J_i = 4\pi k\Sigma\mu_i$$

That is:

$$\int \frac{\partial \phi}{\partial \nu} d\sigma = 4\pi \Sigma \mu_i \quad . \quad . \quad . \quad . \quad . \quad (336)$$

where we use the index i to denote the interior singular points enclosed by the surface σ, whereas ν, as always, denotes the inward normal. This relationship combines the more special equations (324) and (335) in a more general law, and is often called the Gauss's equation, after its discoverer.

We shall consider a few special applications. For two sources of the same intensity the velocity potential becomes:

$$\phi = \mu \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \quad . \quad . \quad . \quad . \quad . \quad (337)$$

Since the plane which bisects the connecting line between the two sources perpendicularly is a plane of symmetry formed entirely by stream-lines, we may imagine them replaced by a fixed partition and so we obtain by means of a result already deduced in § 47 the flow of liquid from a source towards a fixed wall.

If the two sources have opposite but equal intensities, the velocity potential becomes:

$$\phi = \mu \left(-\frac{1}{r_1} + \frac{1}{r_2} \right) \quad . \quad . \quad . \quad . \quad . \quad (338)$$

The liquid then flows from the source 2 to the sink 1—that is, the stream-lines begin in 2 and end in 1. If the two centres approach very close to one another, so that the co-ordinates of 1 are ξ, η, ζ and of 2 are $\xi + \Delta\xi$, $\eta + \Delta\eta$, $\zeta + \Delta\zeta$, then the expression (338) becomes:

$$\phi = \mu \left(\frac{\partial \frac{1}{r}}{\partial \xi} \Delta\xi + \frac{\partial \frac{1}{r}}{\partial \eta} \Delta\eta + \frac{\partial \frac{1}{r}}{\partial \zeta} \Delta\zeta \right)$$

In passing to the limit $\Delta = 0$ we can make μ increase in such a way that the products $\mu \,.\, \Delta\xi$, $\mu \,.\, \Delta\eta$, $\mu \,.\, \Delta\zeta$ remain

finite. If we denote them by μ_ξ, μ_η, μ_ζ, we obtain the following finite expression for the velocity potential:

$$\phi = \mu_\xi \frac{\partial \frac{1}{r}}{\partial \xi} + \mu_\eta \frac{\partial \frac{1}{r}}{\partial \eta} + \mu_\zeta \frac{\partial \frac{1}{r}}{\partial \zeta} . \quad . \quad . \quad (339)$$

A configuration of this kind, which consists of two infinitely near sources, whose intensities are equal, both being infinitely great, but whose signs are opposite, is called a "doublet." The constant vector whose components are μ_ξ, μ_η, μ_ζ is called the "moment" of the doublet, its direction (from the sink to the source) the "axis" of the doublet. The stream-lines run out from the source in the direction of the axis and return in smaller or greater arcs back to the sink. If the source and the sink change places, only the sense of the stream is changed, but not the position of the stream-lines. Since the distance r of the reference point from the singular point contains the co-ordinates of both points only in the form $x - \xi$, $y - \eta$, $z - \zeta$, we may write in place of (339):

$$\phi = - \mu_\xi \frac{\partial \frac{1}{r}}{\partial x} - \mu_\eta \frac{\partial \frac{1}{r}}{\partial y} - \mu_\zeta \frac{\partial \frac{1}{r}}{\partial z} \quad . \quad . \quad (340)$$

and from this we see at once that the function ϕ actually does satisfy the Laplace equation (323).

§ 65. As a further application of the theory that has been developed, we shall also treat a case which may be reduced to the simplified forms here assumed; namely *the uniform motion of a rigid sphere in an incompressible fluid*. For, by the principle of relativity (I, § 59), the laws governing this motion are exactly the same as those which apply to *a stationary sphere in a uniform stream of fluid*, and this case represents a stationary flow of fluid of the kind here considered. So instead of the sphere which is moving with the uniform velocity a in the direction, say, of the z-axis we take a stationary sphere which has its centre at the origin of co-ordinates and

which has a stream of fluid flowing past it on all sides. At all points which are so far distant from the sphere that it no longer exerts an influence—that is, at an infinite distance from the origin—the current has the constant velocity $-a$ in the direction of the z-axis.

The problem is solved if we succeed in building up an expression for the velocity potential ϕ which satisfies Laplace's equation and, besides, satisfies the boundary conditions. The limits of the liquid are $r = R$ (radius of the sphere) and $r = \infty$. For $r = \infty$ we have, according to our above remarks:

$$\phi = az \quad . \quad . \quad . \quad . \quad . \quad (341)$$

and for $r = R$ the normal component of the velocity towards the surface of the sphere is zero, thus:

$$\left(\frac{\partial \phi}{\partial r}\right)_R = 0 \quad . \quad . \quad . \quad . \quad . \quad (342)$$

To satisfy all these conditions we write:

$$\phi = az + \phi' \quad . \quad . \quad . \quad . \quad . \quad (343)$$

The function ϕ' must then satisfy Laplace's equation and also the following boundary conditions:

$$\text{for } r = \infty: \quad \phi' = 0$$

$$\text{for } r = R: \quad \left(\frac{\partial \phi}{\partial r}\right)_R = -a\left(\frac{\partial z}{\partial r}\right)_R = -\frac{az}{R} \quad . \quad (344)$$

Now it can easily be shown that we obtain the solution of this last problem if we set ϕ' equal to the potential function for a doublet—that is:

$$\phi' = \mu \frac{\partial \frac{1}{r}}{\partial z} = -\mu \frac{z}{r^3} \quad . \quad . \quad . \quad . \quad (345)$$

For, in the first place, ϕ' satisfies Laplace's equation; secondly, ϕ' vanishes for $r = \infty$, and thirdly, we get the following constant value for μ from (344):

$$\mu = -\frac{1}{2}R^3 a \quad . \quad . \quad . \quad . \quad . \quad (346)$$

If the differential coefficients $\frac{\partial z}{\partial r}$ and $\frac{\partial r}{\partial z}$ have the same values in equations (344) and (345), namely, the value $\frac{z}{r}$, this is not to be regarded as a contradiction. Only an insufficient understanding of the meaning of these symbols could give rise to this erroneous view. It is, moreover, avoided formally if we write more fully :

$$\left(\frac{\partial z}{\partial r}\right)_{\theta,\phi} = \left(\frac{\partial r}{\partial z}\right)_{x,y} \quad . \quad . \quad . \quad . \quad . \quad (347)$$

That is, on the left-hand side of the equation z is to be imagined as a function of the polar co-ordinates r, θ, ϕ, but on the right-hand side as a function of the rectilinear co-ordinates x, y, z.

By taking into account (345) and (346) we obtain as the solution of the whole problem the required expression for the velocity potential from (343) :

$$\phi = az\left\{1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right\} \quad . \quad . \quad . \quad . \quad . \quad (348)$$

Since ϕ depends only on z and r the flow is symmetrical with respect to the z-axis, and the stream-lines lie in the planes which pass through the z-axis, as is natural.

By (348) the velocity components are :

$$\left.\begin{aligned} u &= -\frac{\partial\phi}{\partial x} = \frac{3}{2}R^3a\cdot\frac{xz}{r^5} \\ v &= -\frac{\partial\phi}{\partial y} = \frac{3}{2}R^3a\cdot\frac{yz}{r^5} \\ w &= -\frac{\partial\phi}{\partial z} = -a\left\{1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\left(1 - \frac{3z^2}{r^2}\right)\right\} \end{aligned}\right\} \cdot \quad (349)$$

To find the stream-lines which lie in any plane which passes through the z-axis we introduce in place of x and y the distance ρ from the z-axis :

$$\rho^2 = x^2 + y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (350)$$

and so :

$$r^2 = \rho^2 + z^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (351)$$

Then the differential equation of the stream-lines runs:

$$d\rho : dz = \frac{\partial \phi}{\partial r} : \frac{\partial \phi}{\partial z}$$

and by (349):

$$d\rho : dz = \frac{3}{2} R^3 a \cdot \frac{\rho z}{r^5} : -a\left\{1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\left(1 - \frac{3z^2}{r^2}\right)\right\}$$

If we here replace z by r and ρ by means of the relationships:

$$z^2 = r^2 - \rho^2 \text{ and } zdz = rdr - \rho d\rho$$

we get, after simplifying:

$$\frac{3R^3 dr}{r^4\left\{1 - \left(\frac{R}{r}\right)^3\right\}} + \frac{2d\rho}{\rho} = 0$$

and hence, by integrating:

$$\log\left\{1 - \left(\frac{R}{r}\right)^3\right\} + 2\log\rho = \text{const.}$$

or:

$$\rho^2 \cdot \left\{1 - \left(\frac{R}{r}\right)^3\right\} = \text{const.}\ (> 0) \quad . \quad . \quad (352)$$

as the equation of the system of stream-lines. This system depends on the radius R of the sphere but not on the velocity a. If we make the constant assume all values from 0 to ∞, we obtain all the stream-lines. When the constant $= \infty$, $\rho = \infty$ and hence also $r = \infty$, the velocity becomes uniform and equal to $-a$. When the constant $= 0$, however, either $\rho = 0$ or $r = R$—that is, the stream-line falls into two different parts, the one of which coincides with the z-axis (outside the sphere), the other lies in the spherical surface. At the transition points $\rho = 0$, $z = \pm R$, the "poles" of the sphere, the stream-line has a right-angled kink. The velocity here becomes equal to zero, and the stream-line divides up into an infinite number of branches, which closely envelop the sphere.

It is interesting to investigate how long a particle of liquid situated on the positive z-axis requires to arrive at the pole $z = + R$ of the sphere. Since $x = 0$, $y = 0$, $r = z$ for the motion of this point, we get for the required time, from (349):

$$t = \int_z^R \frac{dz}{w} = \frac{1}{a}\int_R^z \frac{z^3 dz}{z^3 - R^3} = \infty \quad . \quad . \quad . \quad (353)$$

That is, the time which a particle of fluid requires to move past the sphere increases to an unlimited extent the more closely its orbit lies to the spherical surface.

The velocity with which the liquid flows close past the spherical surface is obtained from (349), if we set $r = R$ in it; the value obtained is:

$$\frac{3}{2}\frac{\rho}{R} \cdot a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (354)$$

It attains its greatest value $\frac{3}{2}a$ in the equatorial plane of the sphere. Moreover, it is perfectly symmetrically disposed over the two hemispheres which are separated by the plane.

Finally we shall inquire into the *pressure* which is exerted by the fluid on the sphere. This will also allow us to calculate the resultant force with which a stream of liquid acts on a stationary sphere or, of course, the resistance which a moving sphere experiences in a stationary fluid.

From (322) we get for the pressure p on the sphere, if we neglect gravity and substitute the value (354) for the velocity:

$$p = \text{const.} - \frac{9}{8}\left(\frac{\rho}{R}\right)^2 a^2 k$$

or, if we denote the pressure in the undisturbed stream of fluid—that is, for the velocity a, by p_0:

$$p = p_0 + \frac{a^2}{2}k\left\{1 - \left(\frac{3}{2}\frac{\rho}{R}\right)^2\right\} \quad . \quad . \quad . \quad (355)$$

It is independent of the direction of the stream, being greatest at the poles of the sphere, smallest in the equatorial plane and symmetrical on both sides of it. If we now combine the pressures $p \, . \, d\sigma$ that act on all the surface-elements of the sphere into a single resultant we can see, without calculation, from the symmetry of these forces for positive and negative values of z, that the resultant vanishes. In other words, a stream of fluid exerts no effect at all on a rigid sphere that is at rest in it, or a sphere in uniform motion experiences no resistance in a fluid which is at rest.

This result is flagrantly contradicted by experience and has therefore long been regarded as a celebrated paradox of hydrodynamics. It is explained by the fact that in the equations here used a term has been omitted which plays an essential part in our problems. This term represents the friction which asserts itself at the surface of contact of the two substances. This friction brings it about that in reality at the surface of the sphere not only the normal component but also the tangential component of the velocity is equal to zero—that is, the fluid adheres completely to the surface of the sphere. We shall follow out this case still further later (§ 80).

§ 66. We now pass on to investigate a further class of particular solutions of Laplace's equation for the velocity potential ϕ—namely, those for which the function ϕ depends only on the two co-ordinates x and y, but not on z, so that:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (356)$$

The problem then reduces to one of two dimensions, which is in many respects simpler. The equi-potential surfaces ϕ = const. are then cylindrical surfaces parallel to the z-axis and may be represented by lines in the xy-plane; these are the equi-potential lines. The system of stream-lines is then represented by the lines orthogonal to these in the same plane.

A very general method of integrating the differential equation (356) depends on the introduction of complex quantities. For it can easily be shown that *every* analytic function of the complex quantity $x + iy = z$ gives rise to a solution of the equation (356) and therefore allows itself to be interpreted as an irrotational stationary motion of an incompressible fluid. Let us denote such a function by w, thus:

$$w = f(x + iy) = f(z) \quad . \quad . \quad . \quad . \quad (357)$$

where the functional expression f may contain arbitrary real or complex constants.

If we then resolve w into its real and imaginary parts:

$$w = \phi + i\psi \quad . \quad . \quad . \quad . \quad . \quad (358)$$

then ϕ and ψ are certain real functions of the real variables x and y. It now follows from (358) that:

$$\frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}, \quad \frac{\partial w}{\partial y} = \frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y} \quad . \quad . \quad (359)$$

On the other hand, we get from (357):

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial x} = \frac{\partial w}{\partial z} \quad . \quad . \quad . \quad . \quad (359a)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial y} = \frac{\partial w}{\partial z} \cdot i$$

hence:

$$\frac{\partial w}{\partial y} = i \cdot \frac{\partial w}{\partial x}$$

and, from (359), by separating the real and the imaginary parts, we get:

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}, \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} \quad . \quad . \quad . \quad . \quad (360)$$

The elimination of ψ gives the equation (356); the elimination of ϕ gives the equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad (361)$$

Moreover, the following relationship holds:

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial y}=0 \quad . \quad . \quad . \quad . \qquad (362)$$

We may therefore interpret ϕ as a velocity potential. The curves ϕ = const. are then the equipotential lines and the corresponding stream-lines are represented by ψ = const., because by (362) the two families of curves are perpendicular to each other. But the functions ϕ and ψ may also change their rôles, so that the latter may be interpreted as a velocity potential and the former as a stream-function.

The nature of the dependence of the complex function w of z may be best illustrated by regarding w and z as those points of two planes, the w-plane and the z-plane, which are characterized by the co-ordinates ϕ and ψ or x and y, respectively. The function $f(z) = w$ then represents a definite correspondence between the two points—that is, the representation of the z-plane on the w-plane and conversely. We may also regard the representation as a deformation of a substance which is uniformly distributed over a plane surface; for this purpose we take ϕ and ψ to denote the co-ordinates of a point of the substance before the deformation, x and y as the co-ordinates of the same point after the deformation, or vice versa. Then precisely the same conclusions may be drawn about the peculiarities of this deformation, as were drawn earlier, in the first chapter, for the more general case of a spatially extended substance. We shall here discuss the most important results.

In the first place, it is easy to see that infinitesimal areas are transformed *linearly* and hence may be transformed into each other by means of a translation, a rotation and a dilatation (§ 11). For it follows from (357) that:

$$dz=\frac{dz}{dw}\cdot dw \quad . \quad . \quad . \quad . \quad . \qquad (363)$$

We now set:

$$\frac{dz}{dw} = \zeta = r(\cos\theta + i\sin\theta) \quad . \quad . \quad . \quad (364)$$

where:

$$r > 0 \quad \text{and } \pi > \theta > -\pi$$

It then follows from (363) if we separate the real and the imaginary parts that:

$$\begin{aligned} dx &= r\cos\theta \,.\, d\phi - r\sin\theta \,.\, d\psi \\ dy &= r\sin\theta \,.\, d\phi + r\cos\theta \,.\, d\psi \end{aligned}$$

If we now keep the two corresponding points w and z fixed and hence also the values of r and θ, and consider only the differentials $d\phi$, $d\psi$, dx, dy as variable co-ordinates, the last two equations give the laws for the representation of regions which are infinitely near w and z; these laws are thus expressed by means of linear relationships. Their geometrical meaning can be immediately recognized by writing:

$$\left.\begin{aligned} \frac{dx}{r} &= \cos\theta \,.\, d\phi - \sin\theta \,.\, d\psi = dx' \\ \frac{dy}{r} &= \sin\theta \,.\, d\phi + \cos\theta \,.\, d\psi = dy' \end{aligned}\right\} \quad . \quad (365)$$

That is, to transform an infinitely near point from the position ($d\phi$, $d\psi$) to the position (dx, dy) we must first perform a simple rotation of angle θ, which causes the point to take the position (dx', dy') and then a uniform dilatation in all directions (end of § 7) in the ratio $r : 1$, which causes all the distances to be magnified in this ratio, whereas all directions, and hence also all angles, remain unchanged. The transformed region is therefore similar to the original region and only rotated with respect to it. This type of representation or transformation by means of a complex function is therefore also said to be "conformal" in its smallest parts.

Hence every conformal representation of the w-plane in the z-plane corresponds to a stationary motion of fluid in the z-plane; and every stream-filament, since it is bounded by two lines $\psi =$ const., forms in the w-

plane a strip which is parallel to the ϕ-axis. So the problem of finding the stationary flow of liquid between any two given fixed boundary lines (walls) resolves itself into the conformal representation of a strip bounded by two parallel lines in the w-plane on the region which lies between these two boundary lines in the z-plane.

The characteristic features of this transformation, the quantities r and θ, shed a vivid light on the nature of the motion of the fluid. For in view of (359a) and (360):

$$\zeta = 1 : \frac{dw}{dz} = 1 : \frac{\partial w}{\partial x} = 1 : \left(\frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}\right)$$

and so we obtain by (364), after separating the real and the imaginary parts:

$$\frac{1}{r^2} = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \quad . \quad . \quad . \quad . \quad (366)$$

$$\cos\theta = r \cdot \frac{\partial \phi}{\partial x}, \quad \sin\theta = r \cdot \frac{\partial \phi}{\partial y} \quad . \quad . \quad (367)$$

So r is the reciprocal value of the velocity and θ (which lies between π and $-\pi$) is the angle which the gradient of the velocity potential (in the direction opposite to the stream-line) makes with the positive x-axis.

As a simple case we consider the flow of a fluid in the space between two fixed straight lines which meet in a sink O at the angle α. (Efflux from an infinitely small orifice at O.) For this it is necessary to represent a strip parallel to the ϕ-axis and of a certain width β in the w-plane conformally on the sectorial section of angle α in the z-plane (Fig. 12). This transformation is represented by the function:

$$z = e^{\frac{\alpha}{\beta} \cdot w} \quad . \quad . \quad . \quad . \quad . \quad (368)$$

or:

$$\left.\begin{aligned} x &= e^{\frac{\alpha}{\beta}\cdot\varphi} \cos\left(\frac{\alpha}{\beta}\psi\right) \\ y &= e^{\frac{\alpha}{\beta}\cdot\varphi} \sin\left(\frac{\alpha}{\beta}\psi\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (369)$$

Actually, if we allow ϕ to decrease from $+\infty$ to $-\infty$, corresponding to the direction of the stream-lines (denoted by arrows in Fig. 12), the point z traverses the one fixed straight line, the x-axis from $x = \infty$ to $x = 0$, for $\psi = 0$, but for $\psi = \beta$ it traverses the other fixed straight line, from infinite distances to O, and for an intervening constant value of ψ it traverses an intervening straight line from infinity to O. Thus the stream-lines in the z-plane are the straight lines which lie within the

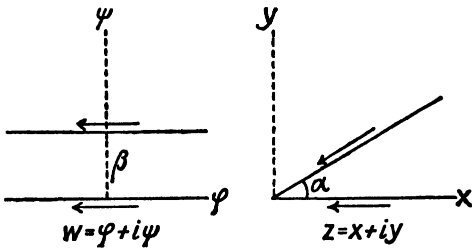

FIG. 12.

angle α and meet at O. If we express ϕ and ψ in terms of x and y, we get from (369):

$$\left.\begin{aligned} \phi &= \frac{\beta}{\alpha}\log\sqrt{x^2+y^2} = \frac{\beta}{\alpha}\log\rho_0 \\ \psi &= \frac{\beta}{\alpha}\tan^{-1}\frac{y}{x} \end{aligned}\right\} \quad . \quad . \quad (370)$$

Here we therefore have the logarithmic potential as a particular solution of the differential equation (356) (cf. I, § 46). The constant β clearly corresponds with the intensity of flow.

For the quantities r and θ we get from (366) and (367):

$$r = \frac{\alpha}{\beta}\cdot\sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\frac{y}{x} \quad . \quad . \quad (371)$$

which agrees with the fact that r denotes the reciprocal of the velocity and θ the direction opposite to that of the stream-line.

§ 67. The method of conformal representation also enables us to master a problem which is not soluble under more general conditions—namely, the representation of fluid-motions that correspond to "free rays": these are streams of fluid that do not occur between fixed walls but in the free air. The problem is much more difficult in this case because the limits of the region of the fluid are not given from the outset, but must first be calculated from the surface conditions which are characteristic for a free ray. For the boundary condition which holds at the surface of a free ray is not only restricted to the condition that this surface must be formed of stream-lines, but it also requires, on account of the principle of action and reaction, that at the surface of the free ray the pressure p of the fluid be equal to the pressure p_0 of the free air—that is, constant.

If we neglect the influence of gravity it follows from (322) that at the surface of a free ray the value of the velocity of flow is constant. Conversely, every stream-line in which the velocity is constant may represent the boundary of a free ray. If the velocity changes at any part of a stream-line while it remains constant at another part, the first part represents the flow along a fixed wall, whereas the second represents the continuation of the flow as a free ray, as is realized, for example, in the case of a fluid which flows out of a tube of arbitrary shape into the free air.

To represent a free ray in the special case of two-dimensional streams here treated it is therefore necessary to find such stream-lines for which not only ψ but also the velocity $\frac{1}{r}$ is constant, and this problem can be solved if we reflect that the lines $\psi =$ const., as already remarked above, denote the straight lines in the w-plane which are parallel to the ϕ-axis, whereas the lines $r =$ const., by (364) denote in the ζ-plane the concentric circular arcs which are described about the origin as centre. So if we succeed in conformally representing a strip in the

w-plane composed of two parallel straight lines on a limited region, partly bounded by a circular arc r = const., in the ζ-plane we get from the functional relationship between w and $\zeta = \frac{dz}{dw}$ a definite functional relationship between z and w—that is, a representation of the strip in the w-plane on the z-plane, which has the peculiarity that on certain boundary lines of the region of the z-plane both ψ = const. and r = const. These lines represent the surfaces of free rays in the motion of the liquid. Helmholtz * was the first to find exact solutions for the problem of the formation of free rays by means of the method here described.

§ 68. We shall now resume our discussion of the general case of an irrotational flow of fluid to shed further light on a question which we referred to briefly in § 61: the question as to whether a stream-line can return into itself or not. We found there that this is impossible if the velocity potential is a one-valued and continuous function of the space-co-ordinates. But it may easily be shown that actually there are irrotational streams which contain closed stream-lines. If, for example, we start out from the simple flow of fluid represented by (370) and recollect that, as was emphasized earlier, the functions ϕ and ψ can exchange their rôles, we obtain the fluid motion:

$$\left.\begin{aligned} \phi &= \frac{\beta}{\alpha}\tan^{-1}\frac{y}{x} \\ \psi &= \frac{\beta}{\alpha}\log\sqrt{x^2+y^2} = \frac{\beta}{\alpha}\log\rho_0 \end{aligned}\right\} \quad . \quad . \quad (372)$$

Here we have an irrotational stationary flow of a fluid which fills the whole xy-plane except for an arbitrary small region which contains the singular point O; in it the stream-lines ψ = const. are concentric circles with the singular point O as centre, and the equi-potential

* H. v. Helmholtz, "Wissenschaftliche Abhandlungen." Leipzig (J. and A. Barth), Vol. I, p. 146, 1882.

lines are the straight lines which start out from O. If we regard any two of these concentric circles as fixed walls, we have a flow in a circular ring-shaped surface. The velocity components are:

$$\left.\begin{aligned} u &= -\frac{\partial\phi}{\partial x} = \frac{\beta}{\alpha}\cdot\frac{y}{\rho_0^2} \\ v &= -\frac{\partial\phi}{\partial y} = -\frac{\beta}{\alpha}\cdot\frac{x}{\rho_0^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (373)$$

and the value of the velocity is:

$$q = \frac{\beta}{\alpha}\cdot\frac{1}{\rho_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (374)$$

Thus whereas the velocity of rotation is everywhere equal to zero permanently, the fluid constantly moves round in a circle. The apparently paradoxical content of this law vanishes if we bear in mind that the liquid does not rotate as a rigid body, but that it undergoes deformations during its motion, because the angular velocity of the circulation becomes smaller and smaller as we recede from the axis of rotation. Hence the angular velocity of the circulation is to be distinguished carefully from the angular velocity of the rotation. For the former is dependent on the radius of curvature of the orbit of a particle of liquid, whereas the latter is dependent on the rotation of the particle, which is quite independent of the former. Nevertheless, we must not imagine that since the velocity of rotation is zero a particle of the fluid always remains parallel to itself. For the components of the velocity of rotation of a particle are not, say, time differential coefficients of angles, which determine the orientation of the particle. If they *were*, then we should certainly have to infer from the vanishing of the velocity of rotation that those angles are constant. Actually, however, all that follows from the vanishing of the velocity of rotation is that those straight lines which at any moment represent the principal axes of dilatation of the particle undergo no change of direction, whereas all

other straight lines in the particle may very well rotate (§ 7). And since at every moment the principal axes of dilatation change, it happens that after a finite time the particle has been able to execute a finite rotation, whereas the velocity of rotation has actually been equal to zero at every moment. Cf. I, § 137 in this connection.

This simple example represented by (372) may also of course be supplemented by other more general cases of irrotational fluid motions with closed stream-lines, in a plane as well as in space, such as the flow in any tube which turns back into itself.

In all such cases we are compelled to conclude that the velocity potential cannot be uniform and continuous. We find this, indeed, confirmed in the expression for ϕ in (372). We either assume the $\tan^{-1}$ as uniform; then it is not continuous but undergoes a sudden change at some point, which can be chosen at random; or we assume it to be continuous; it is then infinitely multiform. If we have to perform calculations with such a function it is found advantageous in most cases to make the *first* choice and to fix on a definite point of discontinuity in order that the result of the calculation may be quite definite. Of course, the multiform or discontinuous character of the velocity potential has no influence at all on the magnitude and direction of the velocity.

§ 69. Since, then, the character of an irrotational fluid motion depends essentially on whether the velocity potential is uniform and continuous or not, the question now arises whether a general criterion can be set up which allows us to decide at the outset whether for a given motion of the fluid the velocity potential is necessarily uniform and continuous. To answer this question we must first enter more closely into the mathematical meaning of the equations (318), which express the absence of vortex motion. Let q be a vector, whose components u, v, w are given within a certain space R as uniform and continuous functions of the space-co-ordinates x, y, z in such a way that the three equations (318) are satisfied.

The following theorem then holds. If we form the integral:

$$J = \int_1^2 q_s \, . \, ds \quad . \quad . \quad . \quad . \quad . \quad (375)$$

from any definite point 1 of the space R to any other definite point 2 of the same space along any arbitrary curve s which lies entirely within R, the integral retains its value for every arbitrary infinitesimal change of the path of integration s. This theorem is proved as follows. From:

$$q_s = u\frac{dx}{ds} + v\frac{dy}{ds} + w\frac{dz}{ds}$$

it follows that:

$$J = \int_1^2 (udx + vdy + wdz) \quad . \quad . \quad . \quad (376)$$

Hence for an infinitely small variation of the curve s:

$$\delta J = \int_1^2 (u \, . \, \delta dx + \delta u \, . \, dx + \ldots) \quad . \quad . \quad (377)$$

Now since $\delta dx = d\delta x$, we get by means of integration by parts, taking into account that the limits of the integral remain unvaried:

$$\int_1^2 u \, . \, \delta dx = -\int_1^2 du \, . \, \delta x$$

$$= -\int_1^2 \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right) . \, \delta x$$

Further, we have:

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

Consequently, if we substitute these and the analogous transformations in (377), we get:

$$\delta J = \int_1^2 \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \cdot (\delta y \, dz - \delta z \, dy) + \quad . \quad . \quad . \quad (378)$$

and, by (318):

$$\delta J = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (379)$$

which was to be proved.

Accordingly, if the value of J is invariant with respect

to an infinitesimal variation of the path of integration, it does not yet follow from this that J has the same value for *all* possible paths of integration. For this conclusion is clearly only justified if from a single path of integration we can, by performing successively infinitesimal—that is, uniform—variations derive all the other paths of integration. Whether this is the case or not depends on the constitution of the space R, within which the vector q is defined. For example, if R has the form of a parallelepiped or an ellipsoid, all curves which run between two definite points inside R can be transformed into each other by continuous variations. Such a space is said to be *simply connected.* In it, therefore, the integral J is entirely independent of the path of integration and is fully determined by the two limits; and hence the velocity potential, which is represented, of course, by J, is one-valued except for the additive constant conditioned by the lower limit.

But if, for example, R has an annular shape, like a tube which returns into itself, it is not possible to transform a curve which connects two points of the tube together, by means of continuous transformations into another curve, which establishes the connection between the two points by traversing the tube in the reverse direction. Such a space, in which not all connecting lines between two points can be transformed into one another continuously, is said to be *multiply connected.* In it the lines connecting two definite points resolve into a number of groups for each of which the integral J can have a different value.

We may also formulate the above described difference between simply and multiply connected spaces in a somewhat different way, which presents certain advantages. It is evident that two different connecting lines between two points can always be combined into a single closed circuit by taking one line to get from the one point to the other and then using the other line as the return path. Now, whether we can transform either line into

the other by continuous variations or not decides whether the closed circuit can be contracted to a single point by a continuous process of contraction or not. Hence we have the theorem: a space is simply or multiply connected according as all the closed lines that lie in it can be continuously contracted to a single point or not. In a parallelepiped or ellipsoid the former is the case; but not in a ring-shaped tube, because a closed line which runs round the ring in a definite sense will meet the side after continued contraction and this will prevent further contraction.

To apply this point of view to the case that interests us here we reflect that the question as to whether the integral J in (375) has the same value for two different paths of integration is equivalent to the question as to whether this integral is equal to zero for the closed circuit composed of the two paths. For the closed integral, that is the integral over the circuit, is the algebraic sum of the two individual integrals, one being taken in the reverse sense. According to the theorem expressed in (379) the closed integral retains its value for any arbitrary continuous contraction of the path of integration, and the points 1 and 2 may also be displaced in this process. If then, as is always possible in a simply connected space, the contraction can be continued to a single point, the length of the path of integration and hence also the value of the integral ultimately become vanishingly small; so that even if the path of integration is arbitrarily extended the integral remains zero, and all the different values of J between two definite points become equal: in other words, the velocity potential is one-valued.

If, however, in the case of a multiply connected space the process of contraction of the closed path of integration encounters a barrier, the integral over such a closed curve can have a value which differs from zero, but which is then the same for all other closed curves that can be derived from it by continuous variation. That is, the

velocity potential is many-valued; in other words, it has a discrete number of values.

§ 70. To avoid the inconvenience which arises from a quantity having many values, we may also transform a multiply connected space into a simply connected space by means of an expedient restriction which can be arbitrarily fixed—namely, by inserting partitions across the space at various points with the condition that the connecting line between two points of the space must not pass through a partition. Then the number of different possible kinds of connecting lines between two points that can be continuously transformed into one another evidently becomes less, and by introducing a sufficient number of such partitions we can arrange that only one kind is left, so that the space becomes simply connected. If a single partition is sufficient to achieve this result, as in the case of a tube that returns into itself, the space is said to be doubly connected: if n such partitions are necessary, it is said to be $(n + 1)$-ply connected.

An example of a space which is more than doubly connected is given by a wheel which has several spokes. If the wheel has four spokes, four partitions are necessary; these partitions can all be imagined to be inserted through the rim at points between two spokes. Hence this space is quintuply connected.

If we imagine the necessary number of partitions to be erected in the manner described, the space becomes simply connected and all the closed curves drawn in it may be contracted to a single point by a continuous process, and the velocity potential is then one-valued. But this gain of one-valuedness is purchased at the cost of loss of continuity. For if we inquire into the difference in the values of the velocity potential at two points, 1 and 2, which are infinitely near each other, but lie on opposite sides of a partition, we have to perform the integration in (375), which serves to calculate this difference not along the infinitely short connecting line, but along the only finite connecting path which is permissible

for this purpose, and since this will give a finite value for the integral, the desired difference is finite—that is, the velocity potential undergoes an abrupt change at the separating partition. But since the path of integration used in this calculation represents a closed curve, and since the value of the integral, according to (379), does not change for any arbitrary continuous variation of this curve, we arrive at the theorem : *the abrupt change in the velocity potential at a surface of discontinuity is the same at all the various points of this surface, and is equal to the value of the integral* (375) *for any closed line which passes through the surface of discontinuity.*

All these theorems are strikingly illustrated in the simple case, discussed in § 68, of the circulation of a fluid in a plane circular ring. Here the stream-lines are closed, the velocity potential is many-valued and the space is doubly connected. In this case a closed stream-line cannot be contracted to a single point, because the small region that contains the singular point and that does not belong to the volume of the fluid is an obstacle to unlimited contraction. But if we imagine a separating partition placed through the fluid, from the singular point to infinity or, in the case where two concentric circular rigid walls are present, from one wall to the other, the fluid space is simply connected and the velocity potential is one-valued but discontinuous. The abrupt change at the surface of discontinuity is the same at all points of this surface and is easily obtained for a closed streamline by calculation from (375); so by (374):

$$\int q \, . \, ds = \frac{\beta}{\alpha} \cdot 2\pi$$

which agrees with the change 2π in the $\tan^{-1}$ in the expression (372) for the velocity potential ϕ.

§ 71. Since a stream-line in a simply connected space cannot return into itself and since it cannot end on a rigid wall, we must infer that *in a simply connected space which is bounded on all sides by rigid walls an irrotational*

motion of an incompressible fluid is altogether impossible. This can actually be proved directly by considering the identity (81). For, on account of the rigid wall, the surface-integral is equal to zero, and on account of incompressibility of the fluid the space-integral on the right-hand side of the equation is equal to zero. Hence ϕ is constant throughout—that is, the fluid is at rest. But in order that (81) should be valid, it is necessary that the velocity potential should be one-valued and continuous. For otherwise the integral:

$$\int_1^2 \frac{\partial \phi}{\partial x} dx$$

would not have the value $\phi_2 - \phi_1$, which was used in (76).

§ 72. To conclude this chapter, we shall deal with a simple example of a non-stationary irrotational motion of an incompressible fluid. For this purpose we shall follow on the case treated in § 63 of the efflux of a heavy fluid out of a tube which is wide above and narrow below, by assuming again that the upper level of the fluid is subject to the constant pressure p_{02} and the orifice of escape is kept at the constant pressure p. We shall not now, however, consider the stationary efflux, but shall inquire into the motion which occurs if the fluid is at rest everywhere at the time $t = 0$.

Concerning the differential equations of the motion, we see that Laplace's equation (323) applies and also the equations (324) to (329), which result from it, in the case of non-stationary motions as well. For it follows from the general equation of continuity (259) in combination with the condition of incompressibility. On the other hand, (322) becomes generalized in conformity with (321) to:

$$p = -\frac{k}{2}(u^2 + v^2 + w^2) - kgz + k\frac{\partial \phi}{\partial t} \quad . \qquad (380)$$

To these there are to be added the boundary conditions at the walls and at the two orifices, as well as the initial condition, for $t = 0$.

All these conditions can be satisfied by setting:

$$\phi = T \, . \, \phi' \quad . \quad . \quad . \quad . \quad . \quad (381)$$

where T is a function of the time t alone, but ϕ' is to be regarded as the known expression for the velocity potential for the case of stationary efflux, specially treated above. For then Laplace's equation is satisfied as well as the boundary conditions at the rigid walls.

In the differential equation (320) for the stream-lines the factor T cancels out entirely; hence the stream-lines at any moment run as in the case of stationary efflux. The only thing that alters with the time is the velocity of flow, and it follows directly from (381), by differentiation, that:

$$q = T \, . \, q' \quad . \quad . \quad . \quad . \quad . \quad (382)$$

That is, the velocity at any point is proportional to the stationary velocity at this point and the time factor T.

If we now again denote the quantities that refer to the upper level of the fluid by the suffix 0, while we leave those that refer to the orifice of escape without a suffix, and neglect f in comparison with f_0, we get from (329), (380), (381) and (382) that:

$$\frac{dT}{dt} = \frac{q'^2}{2(\phi'_0 - \phi')} \, . \, (1 - T^2) \quad . \quad . \quad . \quad (383)$$

By integrating this differential equation, using as an abbreviation the positive constant α thus:

$$\frac{q'^2}{\phi'_0 - \phi'} = \alpha$$

and recollecting that for $t = 0$ $T = 0$ we get:

$$T = \frac{e^{\alpha t} - 1}{e^{\alpha t} + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (384)$$

In conjunction with (382) this gives the velocity of efflux of the fluid at any arbitrary time t. It increases from 0 up to its stationary value in (331), which, strictly speaking, it attains only after an infinite time; but to a certain approximation it attains it the more rapidly the greater it is.

CHAPTER III

VORTEX MOTIONS

§ 73. Let us consider an incompressible fluid which fills a simply connected space bounded by rigid walls on all sides, but which is otherwise arbitrarily great. We then know from § 71 that if no vortices exist in it its velocity is everywhere and for all time zero. But if we now assume that vortices are present somewhere and at some time—that is, that at a certain moment of time the rotational velocity has a given value different from zero at certain places—then it can be proved in this case that the velocity of the whole fluid, inside and outside the vortices, is uniquely determined for the moment of time in question and also for all other times.

We first prove this for the moment in question, t, by showing that the components of the rotational velocity ξ, η, ζ in (59), together with the condition of incompressibility:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad (385)$$

everywhere determine the simultaneous values of the velocity components u, v, w. For if this were not the case there would be still other values of the velocity components, say u', v', w', which would fulfil the same conditions. And then the differences $u' - u = u_0$, $v' - v = v_0$, $w' - w = w_0$, regarded as velocity components, would represent a motion of the fluid in the same simply connected space bounded by rigid walls and this motion would not only satisfy the condition of incompressibility, but would also occur quite irrotationally since ξ_0, η_0, ζ_0 vanishes everywhere for it. But such a

motion is not possible. Consequently u_0, v_0 and w_0 are zero throughout.

Now since the velocity components u, v, w are determinate everywhere, they are also determinate within the vortex filaments, and with them also their space differential coefficients with respect to x, y, z, and hence also, the components of the velocity of deformation. As a result of this, however, the change which a vortex filament undergoes in the course of the element of time dt, including its rotation and its deformation, is completely determined; and since by (317) the product of the vortical velocity ω remains constant in time in the cross-section of the filament, the change that occurs in the velocity ω of the vortex in the time dt is also determined, and so we know the magnitude and the direction, and therefore also the components ξ, η, ζ, of the rotational velocity for the time $t + dt$. But this is sufficient to allow us to apply the same reasoning to the moment of time $t + dt$ as previously to the moment t. In this way the process develops uniquely in that we draw our conclusions from every moment of time to an infinitely near later moment, and so the motion of the whole fluid appears determined for all times.

§ 74. After the results obtained in the preceding section, we shall now carry out the actual calculation of the velocity components u, v, w which correspond to definite vortices given by the values of ξ, η, ζ, and for this purpose we shall assume that the fluid extends to infinity in all directions. This problem is equivalent to that of finding three continuous functions u, v, w which satisfy first the condition for incompressibility (385), and secondly the three equations (59) for given values of ξ, η, ζ.

To satisfy the condition (385) first, we write:

$$u = \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}, \quad v = \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x}, \quad w = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \,. \qquad (386)$$

where we take U, V, W to stand for certain continuous

functions whose first differential coefficients are also continuous; these expressions obviously satisfy (385) identically. Without loss of generality we may impose a certain restriction on the functions U, V, W. For it is clear that if instead of U we write $U + \frac{\partial \psi}{\partial x}$, and instead of V, $V + \frac{\partial \psi}{\partial y}$, and instead of W, $W + \frac{\partial \psi}{\partial z}$, where ψ stands for any arbitrary function, then the equations (386) will give exactly values for u, v, w as before. Hence the function ψ can be fixed arbitrarily without prejudicing the calculation of u, v, w. We shall choose ψ so that:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad (387)$$

It can easily be seen that this can always be attained by appropriately choosing the value of $\Delta\psi$.

This having been assumed, we substitute the expressions (386) in the equations (59) and then obtain:

$$2\xi = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right) - \Delta U \quad . \quad . \quad (388)$$

or by (387):

$$\Delta U = -2\xi, \ \Delta V = -2\eta, \ \Delta W = -2\zeta \quad . \quad (389)$$

These are three Poisson differential equations (I, (132)), whose integrals are, by I, § 45:

$$U = \frac{1}{2\pi}\int \frac{\xi' d\tau'}{r}, \ V = \frac{1}{2\pi}\int \frac{\eta' d\tau'}{r}, \ W = \frac{1}{2\pi}\int \frac{\zeta' d\tau'}{r} \quad . \quad (390)$$

Here r denotes the distance of the reference point x, y, z from any other point x', y', z' of the fluid in which the vortex velocity ξ', η', ζ' exists. The integration is to be performed over all the space-elements $d\tau'$ of the fluid, but of course we may omit all those elements of space in which the vortex velocity is equal to zero. For a fluid completely free of vortices we then again obtain the result of § 71.

The fact that the functions U, V, W that have been determined in this way actually have the property (387)

can be recognized directly by substituting the expressions (390), differentiating them with respect to x, y, z and integrating the resulting integrals by parts in accordance with (79). In the process we also make use of the identities:

$$\frac{\partial r}{\partial x} = -\frac{\partial r}{\partial x'}, \frac{\partial r}{\partial y} = -\frac{\partial r}{\partial y'}, \frac{\partial r}{\partial z} = -\frac{\partial r}{\partial z'}. \quad . \quad (391)$$

as well as of the relationship (315).

If we now substitute the expressions found for U, V, W in (386) we get the velocity components at arbitrary point x, y, z of the fluid, inside or outside the vortex.

$$u = \frac{1}{2\pi}\int\left(\zeta'\frac{\partial\frac{1}{r}}{\partial y} - \eta'\frac{\partial\frac{1}{r}}{\partial z}\right)d\tau', \text{ and so forth} . \quad (392)$$

or:

$$u = \frac{1}{2\pi}\int\left(\eta'\frac{z-z'}{r^3} - \zeta'\frac{y-y'}{r^3}\right)d\tau', \text{ and so forth} \quad (393)$$

or, in vectorial language (I, § 87):

$$\boldsymbol{q} = \frac{1}{2\pi}\int\left[\boldsymbol{o}', \frac{\boldsymbol{r}}{r}\right]\frac{d\tau'}{r^2} \quad . \quad . \quad . \quad . \quad (394)$$

Here $\boldsymbol{q}$ denotes the velocity vector at the reference point, r the distance of the reference point from any fluid element $d\tau'$, at which the vortex velocity $\boldsymbol{o}'$ occurs, and $\frac{\boldsymbol{r}}{r}$ denotes the vector of length unity which is directed from the fluid element $d\tau'$ to the reference point.

The meaning of the equation (394) can be realized most strikingly by reflecting that according to it each vortex element of volume $d\tau'$ contributes a definite amount to the velocity $\boldsymbol{q}$ of the fluid at the reference point. This amount is determined in magnitude and direction by the expression:

$$\frac{1}{2\pi}\left[\boldsymbol{o}', \frac{\boldsymbol{r}}{r}\right]\frac{d\tau'}{r^2} = \delta\boldsymbol{q} \quad . \quad . \quad . \quad . \quad (395)$$

That is, it is proportional to the vortex velocity, inversely proportional to the square of the distance, proportional

to the sine of the angle between the vortex axis and the connecting line r, and it stands perpendicularly on the plane which passes through these two directions. The directions o', r', δq form a right-handed co-ordinate system in the order named, or, in other words, the motion δq occurs in the sense of that direction of the velocity which the particles of the vortex element that are *nearest* the reference point possess when they perform their rotation about the vortex axis, which is assumed at rest.

Since the velocity components u, v, w are given as functions of x, y, z, so also are the differential equations of the stream-lines :

$$u : v : w = dx : dy : dz \quad . \quad . \quad . \quad . \quad (396)$$

which are essentially distinguished from the earlier equations (320) in that here no velocity potential exists in general.

§ 75. Let us take as an example a vortex filament of the form of a thin circular ring. The rotational velocity may be assumed to be so great that its product with the cross-section of the ring, or the moment of the vortex-filament, has a finite value. In the neighbourhood of the ring the flow of the fluid is then of course everywhere free from vortices. Those particles of fluid that lie in the plane of the ring all flow perpendicularly towards this place; those which lie within the ring flow in the sense of rotation of the inner particles, and those which lie outside the ring flow in the direction of rotation of the outer particles of the vortex ring. The stream-lines of the fluid free from vortices pass through the interior of the ring in the corresponding sense, then bend outwards spreading themselves symmetrically in all directions of infinite space, exactly as in the case of a "doublet" (§ 64), and after having passed through the plane of the ring in the region outside the ring in the opposite direction they return from the other side, again approaching each other, into the interior of the ring. Thus the stream-lines all encircle the vortex filament, in the sense of its

direction of rotation. In this process the stream velocity is continuous everywhere, inside and outside the vortex, and also in the transition from the fluid which is in vortex motion to that which is not. The value of the stream velocity is correspondingly greater in the interior of the rings where the stream-filaments become narrower than outside where they widen out. Corresponding to the closed stream-lines there is a many-valued or discontinuous velocity potential and an infinite doubly connected fluid space. Actually such a line does not allow itself to be reduced within the irrotational space to a point by means of continuous contraction.

The vortex ring itself is not in a state of rest, but gives itself a definite velocity; a brief consideration based on the law (395) shows that this velocity is in the direction perpendicular to its plane and in the sense of the stream-lines which pass through its centre.

Now suppose that two of such circular vortex rings are present in the fluid, equally great, of the same moment and parallel to each other, and we shall suppose that the line connecting their centres is perpendicular to their planes and so is a symmetrical axis with respect to the whole process. It is then easy to see what will happen. In the first place, both vortex rings move with their own translational velocity in the direction of the axis of symmetry, in the sense of the stream-lines that pass through them. But, in addition, certain mutual actions will arise between them. For since the equation (394) holds independently of whether the motion of the fluid at the reference point is irrotational or not, the vortex elements of the one ring will be dragged along, so to speak, by the stream-lines generated by the other ring. The consequence of this is that the first ring, which is in advance, will widen, and its velocity will gradually decrease, whereas the second ring, which is following, will become narrower and will advance more and more rapidly until it ultimately catches up with the first ring and is drawn through it. When this has happened the process will

reverse itself, in that now the second ring will again broaden out and will decrease in velocity, while the first narrows itself, until finally the rings are again of equal size and their distance has become the same as at the beginning. The whole series of events then repeats itself, the rôles of the two rings again becoming exchanged, and so it continues indefinitely.

§ 76. We shall carry the calculations for the two-dimensional case a little further, as in § 66. In this case we assume the motion of the fluid to take place parallel to the xy-plane and to depend only on the co-ordinates x and y. The condition of incompressibility (385) then simplifies to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Of the three rotational components in (59) ξ and η vanish, so that we are left only with:

$$\zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad . \quad . \quad . \quad . \quad (397)$$

Hence by (389):

$$U = 0, \ V = 0$$

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = -2\zeta \quad . \quad . \quad . \quad (398)$$

and by (386):

$$u = \frac{\partial W}{\partial y}, \ v = -\frac{\partial W}{\partial x} \quad . \quad . \quad . \quad (399)$$

Thus the vortex lines and the vortex filaments are all always parallel to the z-axis, and since they move parallel to the xy-plane the length of each vortex filament remains constant in time. But since the mass of a vortex filament and, on account of the incompressibility of the fluid, also the volume, do not change with the time, it follows that the cross-section of every vortex filament, and hence by (314) also its rotational velocity ζ, has a value which is invariable in time.

By (396) and (399) the equation of the system of stream-lines:

$$W = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (400)$$

is valid both inside and outside the vortex. Since the stream-lines run parallel to the xy-plane, it is quite immaterial for any motion represented by the above equations whether the fluid stretches along the z-axis to infinity in both directions or whether we imagine the fluid to be enclosed between two arbitrary rigid walls parallel to the xy-plane.

By I, (139) and (140) the two-dimensional Poisson equation (398) is integrated by means of the logarithmic potential:

$$W_{xy} = -\frac{1}{\pi}\int \zeta' \,.\, \log \rho \,.\, d\sigma' \quad . \quad . \quad . \quad (401)$$

where $d\sigma'$ denotes the cross-section of an infinitely thin vortex filament, ζ' the vortex velocity there, ρ the distance of the reference point, and the integral is to be taken over all the vortex filaments. From this we get, by (399), for the velocity components at any point x, y situated inside or outside the vortex the values:

$$\left.\begin{aligned} u &= -\frac{1}{\pi}\int \zeta' \,.\, \frac{y - y'}{\rho^2}\, d\sigma' \\ v &= \frac{1}{\pi}\int \zeta' \,.\, \frac{x - x'}{\rho^2}\, d\sigma' \end{aligned}\right\} \quad . \quad . \quad (401a)$$

which state that every vortex element $d\sigma'$ produces at the reference point a velocity which is inversely proportional to the distance and has the same direction as that of the rotation.

Let us take a single infinitely thin vortex filament, say at the origin of co-ordinates. Then we may assume the rotational velocity to be so great that the moment $\zeta' . d\sigma' = \mu$ has a finite value. The integral in (401) then contracts to a single element and we get:

$$W = -\frac{\mu}{\pi} \,.\, \log \rho \quad . \quad . \quad . \quad . \quad . \quad (402)$$

Hence by (400) the stream-lines are concentric circles,

and the velocity components at any point outside the vortex are, by (399) :

$$u = -\frac{\mu}{\pi}\frac{y}{\rho^2}, \quad v = \frac{\mu}{\pi}\frac{x}{\rho^2} \quad . \quad . \quad . \quad . \quad (403)$$

Except for a constant factor, these equations are the same as the earlier equations (373). There is only the difference that in the earlier case the fluid which circulated irrotationally in concentric circles was bounded by a rigid cylindrical wall which was of no importance, whereas here there is a causal relationship between the moment of the vortex filament which is at rest at the centre and the external flow of the fluid.

For any two such vortex filaments at the points $(x_1\ y_1)$ and (x_2, y_2) we have by (401) :

$$W = -\frac{\mu_1}{\pi}\log\rho_1 - \frac{\mu_2}{\pi}\log\rho_2. \quad . \quad . \quad (404)$$

with the velocities (399) :

$$\left.\begin{aligned} u &= -\frac{\mu_1}{\pi}\frac{y - y_1}{\rho_1^2} - \frac{\mu_2}{\pi}\frac{y - y_2}{\rho_2^2} \\ v &= \quad \frac{\mu_1}{\pi}\frac{x - x_1}{\rho_1^2} + \frac{\mu_2}{\pi}\frac{x - x_2}{\rho_2^2} \end{aligned}\right\} \quad . \quad . \quad (405)$$

By means of these equations we reduce the velocity state of the whole irrotational fluid to the momentary position of the two vortex filaments. The stream-lines are : $W =$ const., and the velocity potential is :

$$\phi = -\frac{\mu_1}{\pi}\tan^{-1}\frac{y - y_1}{x - x_1} - \frac{\mu_2}{\pi}\tan^{-1}\frac{y - y_2}{x - x_2} \quad . \quad (405a)$$

and corresponding to the many-valued character of this expression the irrotational space is triply connected.

As for the motion of the vortex filaments themselves—that is, the changes of the co-ordinates x_1, y_1, x_2, y_2 with the time—we can derive them also from (405). But we must note that a vortex filament is unable to give itself a velocity in the forward direction—this is, of course evident from calculation if we enter into the conditions

that obtain in its interior. In (405) the term corresponding to action of a vortex filament on itself drops out, and we obtain:

$$u_1 = \frac{dx_1}{dt} = -\frac{\mu_2}{\pi}\frac{y_1 - y_2}{\rho_{12}^2}, \quad v_1 = \frac{dy_1}{dt} = \frac{\mu_2}{\pi}\frac{x_1 - x_2}{\rho_{12}^2} \qquad (406)$$

$$u_2 = \frac{dx_2}{dt} = -\frac{\mu_1}{\pi}\frac{y_2 - y_1}{\rho_{12}^2}, \quad v_2 = \frac{dy_2}{dt} = \frac{\mu_1}{\pi}\frac{x_2 - x_1}{\rho_{12}^2} \qquad (407)$$

From these equations we get the following picture of the motion of the two vortex filaments: since:

$$\mu_1 u_1 + \mu_2 u_2 = 0 \text{ and } \mu_1 v_1 + \mu_2 v_2 = 0$$

the point with the co-ordinates:

$$\frac{\mu_1 x_1 + \mu_2 x_2}{\mu_1 + \mu_2} = x_0, \quad \frac{\mu_1 y_1 + \mu_2 y_2}{\mu_1 + \mu_2} = y_0 \; . \quad . \qquad (408)$$

which we may call the "centre of gravity" of the two vortex filaments, remains permanently at rest.

Further:

$$u_1(x_1 - x_2) + v_1(y_1 - y_2) = 0$$
$$u_2(x_1 - x_2) + v_2(y_1 - y_2) = 0$$

That is, the motion of each of the two filaments occurs in a direction perpendicular to the line connecting them. Hence the distance between them and also their distances from the centre of gravity remain constant. Accordingly the filaments rotate with a common angular velocity at a constant distance ρ_{12} from each other about their centre of gravity. If the two rotations are in the same direction, the centre of gravity lies between the two filaments; if they are in opposite directions, the centre of gravity lies outside the two filaments on the side of greater moment. The angular velocity of the rotation is:

$$\frac{\mu_1 + \mu_2}{\pi \rho_{12}^2}$$

and it takes place in the sense of the greater of the two moments.

If the moments are equal to each other and opposite, the centre of gravity recedes to infinity, the angular

velocity of the rotation above referred to becomes zero and the two vortex filaments execute in accordance with (406) and (407) a common translational motion with the velocity $\frac{\mu}{\pi\rho_{12}}$ in the direction of the stream-lines which pass between them, exactly like a single vortex ring (§ 75). Actually we may regard two such "anti-parallel" vortex filaments as components of a single vortex ring of the form of an infinitely long rectangle. The position of the stream-lines is obtained in this special case, by (400) and (404), from the equation :

$$\frac{\rho_1}{\rho_2} = \text{const.} \quad . \quad . \quad . \quad . \quad (409)$$

Hence the stream-lines are also the circles which are symmetrically situated with respect to the line which connects the two vortex filaments, and which divides the distance between them harmonically.

§ 77. Finally we shall consider the simple example of the case of a vortex filament of finite cross-section—namely, a circular vortex filament of radius R and of the rotational velocity ζ which is everywhere uniform and which, as we know, also remains constant in time. Then the integrals in the vortex elements $d\sigma'$ are to be taken over the whole circular surface. The expression for the velocity components at any point of the fluid results either from performing the integration in (401a), or, by a method which is more convenient for us, from calculating the logarithmic potential W of a mass distributed with uniform surface-density $\frac{\zeta}{\pi}$ over the circle, by (401). If we denote the distance of the reference point from the centre of the circle by $\rho_0 = \sqrt{x^2 + y^2}$, then for an internal point ($\rho_0 < R$) we have by I, (145) :

$$W = \frac{\zeta}{2}(R^2 - \rho_0^2) - \zeta R^2 \log R \quad . \quad . \quad (410)$$

whereas for an external point ($\rho_0 < R$) we have by I, (146) :

$$W = -\zeta R^2 \log \rho_0 \quad . \quad . \quad . \quad . \quad (411)$$

Accordingly, by (400) the stream-lines both inside and outside the vortex cylinder are the concentric circles $\rho_0 = \text{const.}$ But whereas outside the cylinder, in the irrotational space, the velocity may be represented, by (399) and (411), as :

$$u = -\zeta R^2 \frac{y}{\rho_0^2}, \quad v = \zeta R^2 \frac{x}{\rho_0^2} \quad . \quad . \quad . \quad (412)$$

which again agree perfectly with (403); the velocity in the interior is given, according to (410), by :

$$u = -\zeta y, \; v = \zeta x \quad . \quad . \quad . \quad . \quad (413)$$

Hence the fluid here rotates with the rotational velocity ζ, without deformation, as a rigid body. Cf. (39). A noteworthy feature is that at the surface of the vortex filament, where the rotational and irrotational fluids border on each other, the velocity remains continuous throughout, since the formulae (412) and (413) transform into each other for $\rho_0 = R$. For this value the velocity attains a maximum value of amount ζR, from which it decreases on both sides to zero.

The pressure p in the external space occupied by irrotational fluid by (322) obeys the following law :

$$p = -\frac{k}{2}\zeta^2 \frac{R^4}{\rho_0^2} + \text{const.} \quad . \quad . \quad . \quad (414)$$

On the other hand, in the interior of the vortex, by (297), it obeys the other law :

$$p = \frac{k}{2}\zeta^2 \rho^2 + \text{const.} \quad . \quad . \quad . \quad . \quad (415)$$

Since by the principle of action and reaction the pressure on the boundary surface is continuous, its value is everywhere determined if it is given at some definite distance—for example, at infinity. If we make the reference point approach from infinity, the pressure decreases as we approach the vortex and continues to decrease to its centre where it is a minimum.

CHAPTER IV

FRICTION

§ 78. THE applications that we have made of the fundamental equations of hydrodynamics in the preceding pages have shown us that these equations, in the form in which they were derived in § 53, enable us to represent a considerable number of fluid motions of different kinds in good agreement with reality. Nevertheless, there are other classes of hydrodynamic phenomena in Nature whose laws are not in conformity with these differential equations. This was borne out most strikingly in the case of the paradoxical theorem of § 65, according to which a sphere which moves through a stationary fluid should experience no resistance to its motion. If we wish to take fuller account of the actual conditions, we must extend the theory appropriately. The question is, in what way ?

The equations of motion (83) and (84) in the first part of the present volume have been derived from the principles of general mechanics which we cannot well alter for our purpose : so we retain these equations. On the other hand, a possibility of modifying the theory suggests itself to us if we reflect that we arrived at our hydrodynamical differential equations (262) only by introducing the hypothesis of "perfect elasticity"—that is, the hypothesis that the pressure depends only on the state of deformation (§ 21). We then obtain for a fluid the simple values (261a) for the six pressure components. We shall now extend this hypothesis by allowing the pressure in an element of the body to vary not only with the state

of deformation, but also with the velocity conditions of the element—that is, with the velocity components u, v, w and their differential coefficients $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots$ Accordingly, we generalize the equations (261a) by writing :

$$\left.\begin{array}{lll} X_x = p + X'_x, & Y_y = p + Y'_y, & Z_z = p + Z'_z \\ Y_z = Y'_z, & Z_x = Z'_x, & X_y = X'_y \end{array}\right\} \quad (416)$$

$$p = f(k) \quad . \quad . \quad . \quad . \quad . \quad (417)$$

and assume that the added pressure components X'_x, Y'_y, . . . depend in some way on the velocity terms referred to. Now it is clear that neither a uniform velocity of translation of the element of the body nor a uniform rotational velocity can have a direct influence on the pressure in it, but only a velocity condition which is connected with a deformation. Hence it follows that of the twelve velocity components (256) only the six components of the deformation velocity x_x, x_y, . . . can enter into the expressions for X'_x, X'_y, . . .; moreover, we shall assume that they can only occur linearly, which will certainly be the case for sufficiently small deformation velocities. We shall also assume them to be homogeneous, because for a vanishingly small deformation velocity the pressure produced by them also vanishes. We shall call the pressure introduced by the new tensor the "friction" of the fluid, and to distinguish it from "external friction" we shall, in particular, call it the "internal friction" or "viscosity." The external friction presents itself when the fluid glides over the surface of contact which it makes with another substance and which has to be taken into account when we set up the boundary conditions.

As soon as the tensor of viscosity is known as a linear homogeneous function of the tensor of deformation velocity, the following laws of motion result from the

general equations (83) and (84) taken in conjunction with (416) and (417):

$$\left.\begin{aligned} \left(X - \frac{d^2x}{dt^2}\right)k &= \frac{\partial p}{\partial x} + \frac{\partial X'_x}{\partial x} + \frac{\partial X'_y}{\partial y} + \frac{\partial X'_z}{\partial z} \\ \left(Y - \frac{d^2y}{dt^2}\right)k &= \frac{\partial p}{\partial y} + \frac{\partial Y'_x}{\partial x} + \frac{\partial Y'_y}{\partial y} + \frac{\partial Y'_z}{\partial z} \\ \left(Z - \frac{d^2z}{dt^2}\right)k &= \frac{\partial p}{\partial x} + \frac{\partial Z'_x}{\partial x} + \frac{\partial Z'_y}{\partial y} + \frac{\partial Z'_z}{\partial z} \end{aligned}\right\} \quad (418)$$

$$Y'_z = Z'_y.\ Z'_x = X'_z,\ X'_y = Y'_x \quad . \quad . \quad (419)$$

To ascertain in what way X'_x, X'_y, . . . depend on x_x, x_y, . . . we have no simpler and no more trustworthy means than that which we used in § 23 to derive the components of elastic pressure in a solid body—namely, the application of the principle of conservation of energy—not, however, as a law of mechanics here, but as a universal physical principle. For friction is not a conservative force (I, § 49); rather, in every process in which friction occurs, a certain amount of mechanical energy becomes transformed into heat. The heat which is generated during an infinitely small interval of time dt in an infinitely small element of the body by friction will be proportional to the quantities dt and $d\tau$: so we immediately set it equal to:

$$W \,.\, dt \,.\, d\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad (420)$$

and we have then to regard W as a finite quantity which is always positive. If we now proceed as earlier in § 23 except that in place of the energy equation (89) we write:

$$d(L + U) + \int W \,.\, dt \,.\, d\tau = A \quad . \quad . \quad (421)$$

and that instead of the equations of motion (83) and (84) for an elastic solid body we use the equations of motion (418) and (419) for a viscous fluid, we finally get instead of (95) the relationship:

$$\int W \,.\, d\tau \,.\, dt + \int (X'_x x_x + X'_y x_y + \ldots) d\tau \,.\, dt = 0 \quad . \quad (422)$$

In the formal respect we must note that in § 23 the symbols u, v, w, x_x, x_y, . . . referred to the *positions*, whereas here, in conformity with (256), they refer to the *velocities*, and hence that in our present nomenclature all these symbols must now be multiplied by the factor dt if they are to denote the same quantities as were early expressed by the differentials du, dv, dw, dx_x, dx_y, ...

Since the equation (422) holds for any volume, however small, we obtain for each point of space:

$$X'_x x_x + Y'_y y_y + Z'_z z_z + Y'_z y_z + Z'_x z_x + X'_y x_y = -\dot{W} \tag{423}$$

Now, as we have seen, the pressure components are homogeneous linear functions of the variables x_x, x_y, . . .; consequently $\dot{W}$ is a homogeneous quadratic function in these quantities. But since on account of the physical meaning of $\dot{W}$ and on account of the isotropic nature of the fluid, the constants of this quantity cannot depend on the choice of the co-ordinate system, we have that by (117) the most general form of $\dot{W}$ is:

$$\dot{W} = \rho(x_x + y_y + z_z)^2 + 2\kappa\left(x_x^2 + y_y^2 + z_z^2 + \frac{y_z^2 + z_x^2 + x_y^2}{2}\right) . \tag{424}$$

where ρ and κ are two positive constants which are characteristic of the viscosity of the fluid.

From these results we now obtain unique expressions for the frictional components. For since the components of the deformation velocity are all independent of one another, the following values for the components of the frictional pressure result from the last two equations:

$$\left.\begin{aligned} X'_x &= -\rho(x_x + y_y + z_z) - 2\kappa x_x, \ . \ . \ . \\ Y'_z &= -\kappa y_z, \ . \ . \ . \end{aligned}\right\} \tag{425}$$

and by substituting in (416) and (418) we get the corresponding result for the values of the total pressure and for the equation of motion.

These formulae contain Stokes's theory of the viscosity of fluids.

For incompressible fluids, which we shall alone consider in the sequel, the equations become considerably simpler, owing to the fact that the velocity of volume dilatation $x_x + y_y + z_z = 0$, and consequently, by (416) and (425):

$$\left.\begin{aligned} X_x &= p - 2\kappa x_x, & Y_y &= p - 2\kappa y_y, & Z_z &= p - 2\kappa z_z \\ Y_z &= -\kappa y_z, & Z_x &= -\kappa z_x, & X_y &= -\kappa x_y \end{aligned}\right\} \quad (426)$$

Hence the vicosity of an incompressible fluid depends only on a single constant κ.

If we wish to formulate completely the laws which govern the motion of a fluid in which viscosity occurs, we have yet to set up the boundary conditions.

If the fluid flows along a stationary rigid wall, no particular condition is required for the normal pressure which acts on the surface, because the resistance of the wall can withstand any pressure at all. When viscosity occurs, a certain condition will, however, hold for the tangential pressure that acts on the surface of the fluid, and the velocity with which the fluid glides along the wall will occur in the condition. If ν again denotes the inward normal to the surface of the fluid, and if X_ν, Y_ν, Z_ν denote the components of the pressure that is exerted on it from without, then the tangential component of this pressure, which, of course, vanishes in the case of a perfectly elastic fluid, will lie, in the case of a viscous fluid, in the direction exactly opposite to that of the flow, and it will be most simple to set its magnitude proportional to the velocity q—that is:

$$X_\nu \cos(\tau x) + Y_\nu \cos(\tau y) + Z_\nu \cos(\tau z) = -\lambda \, . \, q \quad (427)$$

where τ denotes the direction of flow and λ a certain positive constant which depends on the substance of the fluid as well as on that of the wall; λ is called the "coefficient of external friction."

In this method of formulation also the extreme cases $\lambda = 0$ and $\lambda = \infty$ are included. For the former case the tangential pressure vanishes with the external friction;

for the second case, since the tangential pressure remains finite at any rate, the gliding velocity $q = 0$—that is, the fluid "adheres" to the fixed wall. In the framework of the theory here developed the numerical value of the coefficients of internal friction (viscosity) and external friction can be decided only by measurement.

§ 79. We shall consider as our first simple example of the equations that have been set up *the stationary flow of an incompressible fluid through a narrow cylindrical tube of circular cross-section.* Let the pressure at the two ends of the tube be given, the weight of the fluid being neglected.

We choose the axis of the tube as the z-axis and the initial cross-section as the xy-plane: let the pressure be p_0 at this section. If l is the length of the tube, so that $z = l$ is the end cross-section, and if the pressure $p_l < p_0$, the fluid flows in the direction of the positive z-axis. The stream-lines will be parallel to the z-axis, hence:

$$u = 0,\ v = 0 \quad . \quad . \quad . \quad . \quad . \quad (428)$$

It then follows from the condition for the stationary state and from the condition of incompressibility (385) that:

$$\frac{\partial w}{\partial t} = 0, \text{ and } \frac{\partial w}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad (429)$$

so that the following values result from (426) for the pressure-components:

$$\left.\begin{aligned} &X_x = p,\ Y_y = p,\ Z_z = p \\ &Y_z = -\kappa\frac{\partial w}{\partial y},\ Z_x = -\kappa\frac{\partial w}{\partial x}\ X_y = 0 \end{aligned}\right\} . \quad (430)$$

All the acceleration components vanish, so that the equations of motion (83) run:

$$\left.\begin{aligned} &\frac{\partial p}{\partial x} = 0,\ \frac{\partial p}{\partial y} = 0 \\ &\kappa\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \frac{\partial p}{\partial z} = 0 \end{aligned}\right\} \quad . \quad . \quad (431)$$

Hence since p depends only on z, whereas by (429) w depends only on x and y, the last relationship can hold only if:

$$\frac{\partial p}{\partial z} = \kappa\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -c \quad . \quad . \quad . \quad (432)$$

where c, the pressure gradient, denotes a positive constant, whose value is obtained immediately from the fact that for $z = 0$ and $z = l$ the pressure is given—namely:

$$c = \frac{p_0 - p_l}{l} = \frac{\Delta p}{l} \quad . \quad . \quad . \quad . \quad (433)$$

To find w as a function of x and y from (432) we introduce plane polar co-ordinates ρ and ϕ (I, (159)) and note that on account of symmetry w depends only on ρ. We then get:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{d^2 w}{d\rho^2} + \frac{1}{\rho}\frac{dw}{d\rho} = -\frac{c}{\kappa} \quad . \quad . \quad (434)$$

To investigate this differential equation generally we set:

$$\rho\frac{dw}{d\rho} = w' \quad . \quad . \quad . \quad . \quad . \quad . \quad (435)$$

Then:

$$\frac{dw'}{d\rho} = \rho\frac{d^2 w}{d\rho^2} + \frac{dw}{d\rho}$$

and so by (434):

$$\frac{1}{\rho}\frac{dw'}{d\rho} = -\frac{c}{\kappa}$$

Integrated, this gives:

$$w' = -\frac{c}{\kappa}\frac{\rho^2}{2} + A$$

and, by (435), after further integration:

$$w = -\frac{c}{\kappa}\cdot\frac{\rho^2}{4} + A\log\rho + B \quad . \quad . \quad (436)$$

The boundary conditions serve to determine the two constants of integration. For $\rho = 0$, w is finite, and so:

$$A = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (437)$$

Let $\rho = R$ (radius of the tube) for the surface of the fluid. We then have in (427):

$$\cos(\tau x) = 0, \ \cos(\tau y) = 0, \ \cos(\tau z) = 1$$

and, since ν coincides with $-\rho$, by (74) and (430):

$$Z_\nu = \kappa \frac{\partial w}{\partial x} \cdot \frac{x}{\rho} + \kappa \frac{\partial w}{\partial y} \cdot \frac{y}{\rho}$$

Consequently, by (427) we have for $\rho = R$:

$$\frac{\kappa}{R}\left(x \cdot \frac{\partial w}{\partial x} + y \cdot \frac{\partial w}{\partial y}\right) = -\lambda \, . \, w \quad . \quad . \qquad (438)$$

Here we have to set, by (436) and (437):

$$w_R = -\frac{c}{\kappa} \cdot \frac{R^2}{4} + B$$

and:

$$\left(x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y}\right)_R = \left(\rho \cdot \frac{dw}{d\rho}\right)_R = -\frac{c}{\kappa} \cdot \frac{R^2}{2}$$

Accordingly, the boundary condition (438) runs:

$$B = \frac{c}{\kappa} \cdot \frac{R^2}{4} + \frac{c}{\lambda} \cdot \frac{R}{2}$$

and we obtain from (436) as the solution of the problem, the following value for the velocity of flow, after the values of the constants C and B have been inserted:

$$w = \frac{\Delta p}{4\kappa l} \cdot \left(R^2 - \rho^2 + \frac{2\kappa}{\lambda} R\right) . \quad . \quad . \qquad (439)$$

Thus the velocity decreases as we pass from the middle of the tube to the wall, as is natural. In order that it may have a finite value everywhere, however, not only κ, but also λ, must differ from zero, as can easily be seen.

To test the formula (439) that we have derived it is best to measure the volume of fluid V that has flowed

out in a definite time t. This is obtained by integrating over a cross-section of the tube thus:

$$V = \int_0^R \int_0^{2\pi} w \, . \, t \, . \, \rho d\rho \, d\phi \, .$$

so that:

$$V = \frac{\pi}{8} \cdot \frac{\Delta p}{l} \cdot \frac{R^4}{\kappa}\left(1 + \frac{4\kappa}{\lambda R}\right) \cdot t \quad . \quad . \quad (440)$$

According to this the law of efflux is essentially dependent on the ratio of the viscosity constant κ to the constant λ of *external* friction. If this ratio is small compared with the radius R of the tube, the amount of fluid that flows out is proportional to the fourth power of the radius and depends only on the viscosity (internal friction). But if it is great compared with R, the amount that flows out is proportional to the cube of the radius, and depends only on the external friction. In the intermediate cases the law is more complicated, inasmuch as the influences of the two coefficients of friction become superposed on each other.

The experiments of Poiseuille, which were subsequently confirmed, showed that as a rule the first case is realized in practice, so that we may set $\lambda = \infty$ in the last formula, or:

$$V = \frac{\pi}{8} \cdot \frac{\Delta p}{l} \cdot \frac{R^4}{\kappa} \cdot t \quad . \quad . \quad . \quad . \quad (441)$$

From this it further follows by (439) that at the wall of the tube $\rho = R$, $w = 0$, or that the fluid " adheres " to the substance of the wall.

Experience shows, however, that Poiseuille's law (441) holds only for tubes that are sufficiently narrow. For if the radius R of the tube exceeds a certain critical value, the whole solution of the problem here obtained loses its physical meaning. Even the first assumption (428) does not correspond with reality, since the stream-lines actually no longer run parallel to the axis of the tube, but are influenced by the formation of vortices—

a phenomenon which is denoted by the term "turbulence." According to more recent researches, the occurrence of turbulent motions does not contradict the hydrodynamic equations, but is to be explained by the circumstance that the equations for the flow have several solutions, of which the one here developed is the most simple and represents the stable motion that is realized in Nature only in the case of a sufficiently narrow tube. But the exact theory of turbulence presents considerable difficulties to mathematical treatment.

§ 80. The theory of friction that has been developed also allows us again to attack with better success a problem which we previously (§ 65) found to be insoluble: the calculation of the resistance which a moving sphere experiences in a stationary fluid. We inquire into the magnitude of the total pressure which an incompressible fluid at rest exerts on a sphere of radius R moving with a constant velocity a in a definite direction—say the z-axis. In this case we shall also assume the fluid to be heavy and of density k. Instead of the moving sphere in the stationary fluid we again use the principle of relativity and assume that we have *a sphere at rest*, with its centre at the origin of co-ordinates, *in a uniform stream of fluid*—that is, in a stationary flow of fluid, whose velocity at infinite distance from the sphere $r = \infty$ is:

$$u = 0, \; v = 0, \; w = -a \quad . \quad . \quad . \quad (442)$$

We use as our other boundary condition that found in the preceding paragraph—namely, that at the surface of the sphere, that is, for $r = R$, the fluid adheres to the sphere, so that:

$$u = 0, \; v = 0, \; w = 0 \quad . \quad . \quad . \quad (443)$$

The equations of motion (83) with the values (426) for the pressure components and the condition of incompressibility (385) serve as the differential equations for determining u, v, w and p as functions of x, y, z. In general,

the equations of motion are not linear, since for the acceleration we have:

$$\frac{d^2x}{dt^2} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot u + \frac{\partial u}{\partial y} \cdot v + \frac{\partial u}{\partial z} \cdot w, \ldots \quad . \quad (444)$$

Hence we shall simplify the problem still further by introducing the assumption that the velocities u, v, w are sufficiently small to allow us to neglect the quadratic terms on the right-hand side of (444). The equations (83) then become:

$$\left(X - \frac{\partial u}{\partial t}\right)k - \frac{\partial X_x}{\partial x} - \frac{\partial X_y}{\partial y} - \frac{\partial X_z}{\partial z} = 0 \ldots$$

or, if we substitute the expressions (426) for the pressure components and take into account the condition of incompressibility (385):

$$\left(X - \frac{\partial u}{\partial t}\right)k - \frac{\partial p}{\partial x} + \kappa \,.\, \Delta u = 0, \ldots \quad . \quad (445)$$

and further, since the flow is stationary, and $X = 0$, $Y = 0$, $Z = -g$:

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \kappa \,.\, \Delta u \\ \frac{\partial p}{\partial y} &= \kappa \,.\, \Delta v \\ \frac{\partial p}{\partial z} &= -kg + \kappa \,.\, \Delta w \end{aligned}\right\} \quad . \quad . \quad . \quad (446)$$

We may eliminate p from these three equations, and we then obtain:

$$\Delta\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0, \ \Delta\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0, \ \Delta\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \quad . \quad (447)$$

The whole problem now falls into two distinct parts: in the first place, we have to determine the velocity components u, v, w from (385) and (447), using the boundary conditions (442) and (443), in the second place, we have to determine the pressure p from (446) and the total

action on the sphere which is the resultant of the pressure components.

Concerning the velocity it is clear, on account of the boundary condition, that it can have no potential here. We therefore follow on the equations of § 65, which we generalize slightly, and make the following assumption:

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} + u' \\ v &= -\frac{\partial \phi}{\partial y} + v' \\ w &= -\frac{\partial \phi}{\partial z} + w' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (448)$$

$$\phi = a \, . \, z + b \cdot \frac{\partial \frac{1}{r}}{\partial z} + c \cdot \frac{\partial r}{\partial z} \quad . \quad . \quad . \quad (449)$$

where u', v', w' denote three new functions with simpler properties, and a, b, c are constants. The boundary condition (442) is fulfilled if u', v', w' vanish at infinity. The condition for incompressibility (385) demands that:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = \Delta \phi \quad . \quad . \quad . \quad (450)$$

If we now consider that, by (449):

$$\Delta \phi = c \, . \, \Delta \left(\frac{\partial r}{\partial z} \right) = c \cdot \frac{\partial (\Delta r)}{\partial z}$$

and that:

$$\Delta r = \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r} \quad . \quad . \quad . \quad (451$$

then (450) becomes:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 2c \frac{\partial \frac{1}{r}}{\partial z}$$

To satisfy this equation we set:

$$u' = 0, \; v' = 0, \; w' = \frac{2c}{r}$$

and so we obtain from (448) that:

$$\left.\begin{aligned} u &= -\frac{\partial\phi}{\partial x} \\ v &= -\frac{\partial\phi}{\partial y} \\ w &= -\frac{\partial\phi}{\partial z} + \frac{2c}{r} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (452)$$

These expressions also allow all the other conditions to be satisfied. First we see easily that the equations (447) are satisfied identically. There remain the boundary conditions (443), which give for $r = R$:

$$0 = b\frac{\partial^2 \frac{1}{r}}{\partial x \partial z} + c\frac{\partial^2 r}{\partial x \partial z} = \frac{3bxz}{R^5} - \frac{cxz}{R^3}$$

$$0 = b\frac{\partial^2 \frac{1}{r}}{\partial y \partial z} + c\frac{\partial^2 r}{\partial y \partial z} = \frac{3byz}{R^5} - \frac{cyz}{R^3}$$

$$0 = -a - b\frac{\partial^2 \frac{1}{r}}{\partial z^2} - c\frac{\partial^2 r}{\partial z^2} + \frac{2c}{R}$$

$$= -a - b\left(\frac{3z^2}{R^5} - \frac{1}{R^3}\right) + c\left(\frac{z^2}{R^3} + \frac{1}{R}\right)$$

whence it follows that:

$$b = \frac{aR^3}{4},\ c = \frac{3}{4}Ra \quad . \quad . \quad . \quad . \quad (453)$$

so that u, v, w are now fully determined.

The second part of the calculation concerns the calculation of the pressure. If we use the values obtained for u, v, w the equations (446) give on being integrated:

$$p = -kgz - \kappa\,.\,\Delta\phi + p_0$$

where p_0 denotes the pressure of the stream of fluid which is uninfluenced by the sphere in the plane

$z = 0$; by substituting the expression (449) for ϕ we get:

$$p = -kgz + 2\kappa c \cdot \frac{z}{r^3} + p_0 \quad . \quad . \quad . \quad (454)$$

The total pressure which the fluid in flow exerts on the sphere at rest is the resultant of all the pressures that act on the surface elements $d\sigma$ of the sphere whose inward normal is $\nu = -r$. Hence since they clearly reduce to their z-component, we get by (74):

$$\int Z_\nu d\sigma = \int (Z_x \cos(\nu x) + Z_y \cos(\nu y) + Z_z \cos(\nu z))\, d\sigma \quad (455)$$

into which we have now to insert the values of the pressure components from (426). We must then integrate over the whole surface of the sphere, which is best done by introducing polar co-ordinates: $d\sigma = R^2 \sin\theta d\theta d\phi$. The direct calculation is found to be a little cumbersome. It may be considerably simplified by the following consideration.

By (83) we have for our case in the whole interior of the fluid:

$$kg + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = 0$$

For the acceleration $\frac{d^2z}{dt^2}$ vanishes here, as we saw above from the equations (444).

If we now integrate this equation over the space between the surface of the fixed sphere and any concentric spherical surface of radius $r > R$, using the relation (78) in the process, we find that the integral (455) when taken over the spherical surface of radius R is equal to the same integral taken over the spherical surface of radius r plus the term:

$$-kg(V_r - V_R) \quad . \quad . \quad . \quad . \quad (456)$$

where V_r and V_R denote the volumes of the two spheres. Now, since the choice of r is quite arbitrary, we assume r to be infinitely great, and in this way considerably simplify our expressions.

We have for $r = \infty$, by (452) and (449):

$$z_x = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = -\frac{6cxz^2}{r^5}$$

$$z_y = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = -\frac{6cyz^2}{r^5}$$

$$z_z = \frac{\partial w}{\partial z} = \frac{cz}{r^3}\left(1 - \frac{3z^2}{r^2}\right)$$

Consequently, by (426), we get if we take into account (454):

$$Z_x = \frac{6\kappa cxz^2}{r^5}, \quad Z_y = \frac{6\kappa cyz^2}{r^5}$$

$$Z_z = p_0 - kgz + \frac{6\kappa cz^3}{r^5}$$

If we substitute these values in (455) and integrate over all the elements $d\sigma = r^2 \sin\theta d\theta d\phi$ of the spherical surface r whose inward normal is $\nu = -r$, we obtain the expression:

$$-8\pi\kappa c + kg \,.\, V_r$$

and if (456) is added to this we get for the required pressure on the sphere the value:

$$kgV_R - 8\pi\kappa c$$

or, by (453):

$$kgV_R - 6\pi\kappa Ra \quad . \quad . \quad . \quad . \quad (457)$$

The first term corresponds to the upthrust (I, § 114) caused by the weight G of the fluid which has been displaced by the sphere; the second term corresponds to the frictional resistance; it of course acts in the direction of flow of the fluid (442).

If we consider the sphere to be freely movable and heavy, of weight G_0, and if we inquire into the force F which we must allow to act on it in order that it should remain at rest, we get for the value of F, considered positive in the upward direction:

$$F = G_0 - G + 6\pi\kappa Ra \quad . \quad . \quad . \quad (458)$$

This is at the same time the force which must be exerted on the sphere in the upward direction in order that it shall move along with the constant velocity $+a$ in the fluid at rest. Conversely, this formula of Stokes gives us the value a of the stationary (or steady) velocity with which the sphere moves through the fluid when a constant force F acts on it. If, for example, the sphere falls only by its own weight, then the velocity is :

$$a = -\frac{G_0 - G}{6\pi\kappa R} \quad . \quad . \quad . \quad . \quad . \quad (459)$$

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